Spectral sequence calculation for unstable v_n-periodic homotopy groups of spheres

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Abstract

Besides reviewing historical development and prerequisites, we present the main ingredients needed for calculations with spectral sequences for this purpose. These include the Bousfield-Kuhn functor, the Goodwillie tower and so on. We then outline current approaches to running such spectral sequences from cohomology to homotopy, as well as indicate specific questions to investigate.

Contents

1 Introduction

1.1 The history of v-periodic homotopy group

In the E_2-page of the Adams spectral sequence of the low stems, some structures appear periodically at the top of the table.[1] This naturally raises a question. Does a periodic structure appear in the homotopy groups of spheres?

In [14], Mahowald and Davis use the self-map to construct the v-periodic element in stable homotopy group:

Definition 1.1. Let X be a finite complex. A periodic operator is $v \in [\Sigma^i X, X]$ such that $v^k \neq 0 \in [\Sigma^{ik} X, X]$ for all $k > 0$. A class $\alpha \in [X, Z]$ is v-periodic if $\alpha \circ v^k \neq 0$ for all k. A class $\beta \in [S^j, W]$ is v-periodic if, for some skeleton $X^{(t-1)}$ of X, β can be decomposed as:

$$S^t \xrightarrow{i} X/X^{(t-1)} \xrightarrow{\bar{\beta}} \Sigma^{t-j} W. \tag{1}$$

and for all such β and all $k \geq 0$,

$$\Sigma^{ik} X \xrightarrow{v^k} X \xrightarrow{p} X/X^{(t-1)} \xrightarrow{\bar{\beta}} \Sigma^{t-j} W. \tag{2}$$

is essential. Here Z, W are spectra, and the corresponding group are in the meaning of stable maps.

[1]The study of the periodic phenomenon in the classical Adams spectral sequence can be found in [24], this periodic property also induces a periodic property in the homotopy group of the sphere spectrum.

With the above definition, we can identify these periodic elements in the homotopy group. In particular, a v-periodic element of $\pi_*(S^0)$ gives rise to an infinite family of nonzero elements of $\pi_*(S^0)$ by choosing for each k the first cell on which $\overline{\beta p v^k}$ is essential.

Naturally, we are wondering whether the choice of X and v influence the v-periodic elements in a $\pi_*(Z)$ or not, as well as if X has to satisfy some restrictions to support a self-map. To answer these questions, we need to introduce the chromatic perspective. Roughly speaking, it gives us a filtration of self-maps.

The study of chromatic homotopy theory gives us more information about the self-map. In [17], Devinatz-Hopkins-Smith proved the nilpotence theorem which describes a restriction for X to support a permanent self-map f:

Theorem 1.1. *Let MU_* be the complex bordism theory. There is a homology theory MU_* such that a self-map f of a finite CW complex X is stably nilpotent if and only if some iterates of $\overline{MU}_*(f)$ are trivial. The remaining map is periodic.*

However, the MU is too big. As a result, we always deal with some "localization" versions of it, such as BP, $E(n)$ and $K(n)$, to simplify the problem. In particular, the $K(n)$ can detect a specific part of those permanent self-maps by the periodic theorem proved by Hopkins-Smith in [19]:

Theorem 1.2. *Let X and Y be p-local finite CW-complexes of type n for n finite. The type of a p-local space X can be defined as the smallest integer such that $\overline{K(n)}_*(X)$ is nontrivial.*

- *There is a self-map $f : \Sigma^{d+i}X \to \Sigma^d X$ for some $i \geq 0$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is trivial for $m > n$. When $n = 0$ then $d = 0$, and when $n > 0$ then d is a multiple of $2p^n - 2$. We called this self-map a v_n self-map. What's more, there's an analogue when X is a spectrum.*
- *Suppose $h : X \to Y$ is continuous and both of them have already been suspended enough times to be the target of a v_n-self-map. Let $g : \Sigma^e Y \to Y$ be a v_n-self-map. Then there are positive integers i and j with $di = ej$ such that the following diagram commutes up to homotopy. (The "uniqueness" of v_n map.)*

$$\begin{array}{ccc} \Sigma^{di}X & \xrightarrow{\Sigma^{di}h} & \Sigma^{di}Y \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle g^j} \\ X & \xrightarrow{h} & Y \end{array}$$

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Now we can give an explicit definition of the periodic part in the stable homotopy group.

Definition 1.2. the v_n-homotopy group (with coefficient V) for a spectrum X and a spectrum V which supports a v_n-self-map v:

$$v_n^{-1}\pi_*(X;V) := v^{-1}[\Sigma^* V, X]_{Sp}. \tag{3}$$

We can prove that the choice of v doesn't influence the v_n-homotopy group of X.[28]

1.2 Bousfield localization and stable v_n-periodic homotopy group

If we use the above definition directly, we will find that determining whether a map $f : X \to Y$ induce a v_n-periodic homotopy isomorphism is hard. So we hope to find a simple method to judge this. Recall that in rational homotopy theory and p-adic homotopy theory, the isomorphism between homology groups can induce isomorphism between homotopy groups. Since $K(0)_*$ is the rational homology theory, we have a natural conjecture: Is the $K(n)_*$-isomorphism decides the $v_n^{-1}\pi_*$-isomorphism? If not, is there any other spectrum that can decide it?

Actually, $v_n^{-1}\pi_*$-isomorphism is equivalent to $T(n)_*$-isomorphism. Here the spectrum $T(n)$ is defined as:

Definition 1.3. For a type n p-local spectrum V with a v_n-self-map v,

$$T(n) = v_n^{-1}V := hocolim(V \xrightarrow{v} \Sigma^{-k}V \xrightarrow{v} \Sigma^{-2k}V \xrightarrow{v} \cdots). \tag{4}$$

This spectrum is independent of the choices of V and v in the meaning of homotopy equivalence due to the class invariance theorem.

Here, the class invariance theorem is

Theorem 1.3 (Class Invariance Theorem)**.** *Let X and Y be p-local finite CW-complexes of types m and n. Then $\langle X\rangle = \langle Y\rangle$ if and only if $m = n$, and $\langle X\rangle > \langle Y\rangle$ if and only if $m > n$.*

To reveal the relationship between $T(n)$ and $K(n)$, we need to introduce the Bousfield localization and the Bousfield equivalence. Aiming at calculating $K(n)_*(X)$ as well as other generalized (co)homology theories, Bousfield localization was developed in [10] and [9]. This tool enables us to simplify the X into $L_E X$, an E-local spectrum, without changing $E_*(X)$ for any given homology theory E_*. Bousfield localization can be done functorially.

Bousfield equivalence is designed to classify these localization functors. For two different spectra A and B, their Bousfield equivalence class $\langle A\rangle = \langle B\rangle$ if and only if $L_A X \simeq L_B X$ for any spectrum X.

There is only one thing we need to check: is $\langle K(n)\rangle = \langle T(n)\rangle$ in general? That is the telescope conjecture. In [25] and [23], the case of $n = 1$ was proved. For $n \geq 2$, we believe that this conjecture would fail at an early time. But the disproof was not completed until the 6th, June, 2023.[1] However, we have few tools to calculate $T(n)_*(X)$. As a result, we just use $T(n)_*$ for some abstract proof. If we aim at calculating the v_n-periodic homotopy group, we have to consider $L_{K(n)}X$[2] instead of $L_{T(n)}X$ although some classes of v_n-periodic homotopy group may be killed in the $K(n)$-localization.

1.3 Bousfield-Kuhn functor and unstable v_n-periodic homotopy group

Now, we can move our steps to the unstable range. The periodicity theorem implies that a finite type n complex V also admits a v_n-self map:

$$v : \Sigma^{k(N_0+1)}V \to \Sigma^{kN_0}V \tag{5}$$

for some $N_0 \gg 0$. For any $X \in Top_*$, the unstable v_n-periodic homotopy group (with coefficient V) can be defined as [8]:

$$v_n^{-1}\pi_*(X; V) := v^{-1}[\Sigma^* V, X]_{Top_*} \tag{6}$$

for $n > 0$. Although this definition only makes sense for $* \gg 0$, the k-periodicity ensures that it can be defined on any $* \in \mathbb{Z}$.

Since we only have enough tools to compute homotopy groups in the stable range, we need to pull the unstable v_n-periodic homotopy group back to the stable range. The tool we use is the Bousfield-Kuhn functor Φ_n. It is a functor from Top_* to Sp (actually $Sp_{T(n)}$) which allow us to calculate $v_n^{-1}\pi_*(X)$ by $\pi_*(\Phi_n X)$. We can prove that the definition of unstable v_n-periodic homotopy group is compatible with the stable one. The definition and properties of this functor will be introduced explicitly in the following section. We will only show some important properties of this functor in this section.

Proposition 1.4. •

- Φ_n *preserves fiber sequences.*
- $v_n^{-1}\pi_*(X; V) = [\Sigma^* V, \Phi_n(X)]_{Sp}$
- *If Z is a spectrum, then* $\Phi_n\Omega^\infty Z = L_{T(n)}Z$.

[1] https://www.uio.no/studier/emner/matnat/math/MAT9580/v23/beskjeder/disproof-of-the-telescope-conjecture.html

[2] This corresponds to the chromatic homotopy theory.

1.4 The Goodwillie tower

Now we need to find some methods to calculate the $\pi_*(\Phi_n X)$, or its $K(n)$-localization. The $K(n)$-localization is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$. The tool we need is the Goodwillie tower.

To explain it, we need some ideas in the rational homotopy theory. In rational homotopy theory, the information of rational homotopy type can be encoded by a rational differential graded Lie algebra $\mathcal{L}(X)$ and a rational cocommutative differential graded coalgebra $\mathcal{C}(X)$. Suillivan connected them by the minimal model $\Lambda(X)$[29] with the following equation.

$$\pi_*(X)_{\mathbb{Q}} \cong DQ\Lambda(X) \tag{7}$$

Unstable p-adic homotopy type of a simply connected finite type space is similarly encoded in its $\overline{\mathbb{F}_p}$-valued singular cochains. However, the p-adic analogue of the above equation fails. As a result, people discovered other localization of unstable homotopy groups for which the analogue of the above equation holds. That leads to the unstable chromatic homotopy theory. So there is a tight connection between rational homotopy and v_n-periodic homotopy groups.

Now we talk more about the Lie algebra structure. The algebra structure of $\pi_*(X)$ is decided by the homotopy group of X as well as the Lie algebra structure decided by the Whitehead product. In rational homotopy theory, the structure is simplified to the Lie algebra structure only. What's more, Quillen's work on rational homotopy theory reveals that:

Theorem 1.5. *There is an equivalence of homotopy theories:*
$\{$*Simply connected pointed rational spaces*$\} \xrightarrow{\sim} \{$*Connected differential graded Lie algebras over* $\mathbb{Q}\,\}$
Moreover, if X is a simply connected pointed rational space which corresponds to a Lie algebra g_* *under this equivalence, then the Lie algebra* $(\pi_{*+1}(X), [\bullet, \bullet])$ *can be identified with the homology of* g_*

The lower central series filtration of g_*

$$\cdots \subseteq g_*^{(4)} \subseteq g_*^{(3)} \subseteq g_*^{(2)} \subseteq g_*^{(1)} \tag{8}$$

decides a tower:

$$\cdots X_4 \to X_3 \to X_2 \to X_1 \to X_0 \simeq * \tag{9}$$

This tower is useful because we can use this to calculate $\pi_*(F_n)$, where F_n is the fiber of $X_n \to X_{n-1}$. The Goodwillie tower is a refinement of this picture which works at the integral level.

For a fixed functor F form some category to Spectra, the Goodwillie tower can be represented as a tower

$$\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X). \tag{10}$$

For each $P_i(X)$, we have a map from X to $P_i(X)$ and they satisfy the natural commutative diagram. The fiber of $P_n(X) \to P_{n-1}(X)$ is called $D_n(X)$, which is an infinite loop space $\Omega^\infty((\Sigma^\infty X)^n \wedge \mathcal{O}(n))_{h\Sigma_n}$, where $\mathcal{O}(n)$ can be explicit defined.[1] We have some methods to calculate its homotopy group.

The Goodwillie tower has a commutative property with the Bousfield-Kuhn functor with some minor restrictions. That is,

$$P_k\Phi_n \simeq \Phi_n P_k Id. \tag{11}$$

It is also true for $\Phi_{K(n)}$. So we just need to study the Goodwillie tower of Id to understand the properties of Φ_n.

1.5 The method of the explicit calculation

Now, we can introduce the known paths of calculating $\pi_*(\Phi_{K(n)}(X))$. There are two possible approaches I know to calculate these groups. One is introduced in [30] by Wang, and another one is introduced in [7] by Behrens and Rezk. These two approaches share a similar frame, but the details of the calculation are different. I will introduce the approach of Wang first.

By considering $E(n)$-Adams spectral sequence of $\Phi_{K(n)}(X)$, we have

$$Ext_{E_{n*}E_n}(E_{n*}, E_n(\Phi_{K(n)}(X))) \Rightarrow \pi_*(\Phi_{K(n)}(X)). \tag{12}$$

Here the E_n means E_n-homology. Since $\Phi_{K(n)}(X)$ is $K(n)$-local, according the Appendix A of [16], we can transform it into the following form:

$$H_c^*(G_n, E_n(\Phi_{K(n)}(X))) \Rightarrow \pi_*(\Phi_{K(n)}(X)). \tag{13}$$

Here G_n is the Morava stabilizer group. We call the original form $K(n)$-local $E(n)$-Adams spectral sequence, and the second form as a special case of homotopy fixed point spectral sequence.[2]

Then, we need to calculate the E_n-homology of $\Phi_{K(n)}(X)$. We can give a resolution of $\Phi_{K(n)}(X)$ by Goodwillie tower. By acting E_{n*} on the Goodwillie tower, we can get $E_n(\Phi_{K(n)}(X))$ by the Atiyah-Hirzebruch spectral sequence if we know $E_n(D_k(\Phi_{K(n)}(X)))$ for each k as well as the attaching map of Goodwillie tower. The differentials of that spectral sequence can be calculated by representing those generators in $H_c^*(G_n, \mathbb{F}_p)$.

Here, we need to consider X as S^m where m is odd because we know enough information of $D_k(\Phi_{K(n)}(S^m))$. In this situation, the attaching map of Goodwillie tower is decided by the

[1] $\mathcal{O}(n)$ is the $\partial_k F$, the k^{th} derivative of F, where F is a functor.

[2] Some papers directly use the homotopy fixed point spectral sequence to describe this, but I prefer to treat it as a special case of the Adams-Novikov spectral sequence.

James-Hopf map[1], and $D_k(\Phi_{K(n)}(S^m))$ is homotopy equivalence to some spectrum related to the Steinberg summand of $B\mathbb{F}_p^n$. That is,

$$D_{p^t}S^k \simeq \Omega^\infty\Sigma^{k-t}L(t)_k \tag{14}$$

while $D_mS^k \simeq *$ for $m \neq p^t, t \in \mathbb{N}$. Here the $L(t)_k$ is the Steinberg summand, which we will introduce later.

Since we can get $E_{n*}(X)$ by $BP_*(X) \otimes_{BP_*} E_{n*}$, the remaining work is calculating the BP-homology of $L(t)_k$. The E_2-page of Adams spectral sequence of $BP_*(L(t))$ is calculated in [20] if we know the ordinary cohomology of $L(K)$ and it can be calculated by analysing the base of ordinary cohomology of $L(k)$ represented by $\beta^{\epsilon_1}P^{i_1}\beta^{\epsilon_2}P^{i_2}\beta^{\epsilon_3}P^{i_3}\cdots$ admissible. However, if we need the full comodule structure of it, we need the v_n-hidden extension in this spectral sequence.

As a result, we need the BP-cohomology of $L(t)$, which can be calculated by Koszul complex [2] $BP^*(L(t)_k)$ can be describe by $BP^*(L(t))$ and Dickson-Mui generators.[3]

Then, using this E_n-homology as an input, we get the E_2 page of the above spectral sequence. In particular, for $n = 2$ and $p \geq 5$, there is no possible nontrivial differential in the Adams-Novikov spectral sequence, so we get its unstable v_n-periodic homotopy group.[30]

Another approach uses a similar spectral sequence but it calculates $E_n(S^m)$ in a very different way. In [7], Behrens and Rezk constructed a natural transformation from pointed spaces to $K(n)$-local spectra called the comparison map.

$$C^{S_K} : \Phi_{K(n)}(X) \to TAQ_{S_K}(S_K^{X_+}) \tag{15}$$

This transformation relates $\Phi_{K(n)}(X)$ to the topological Andre-Quillen cohomology. For X is an odd sphere, the comparison map is an equivalence. In addition, those spaces for which the comparison map is an equivalence are called "$\Phi_{K(n)}$-good". Some works have been done to study these spaces such as [6]. But in this article, we will not discuss this.

Ching's work [13] shows that $TAQ_{S_K}(S_K^{X_+})$ has the structure of an algebra over the operad formed by Goodwillie derivatives $\partial_*(Id)$. This can be regarded as a topological analogue of the Lie operad. As a result, we can see $TAQ_{S_K}(S_K^{X_+})$ as a Lie algebra model for the unstable v_n-periodic homotopy type of X (or in a short way, an analogue of $\mathcal{L}(X)$).

Since Dyer-Lashof algebra Δ^q can be used to construct a form of Andre-Quillen cohomology,

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[1] Only for spheres.

[2] Koszul complex can be seen as a special type of bar construction. Behrens and Rezk show that the information of $L(n)$ is encoded by some Koszul complex in their paper.

[3] The $L(t)_k$ can be seen as the result of unstable filtration for k odd. This filtration can be defined by the powers of D_n, the n^{th} Dickson-Mui generator.

we can relate the Andre-Quillen cohomology with the Koszul resolution of Δ^q. It was finished by constructing a bar construction model for Kuhn's filtration on topological Andre-Quillen cohomology, in which layers of this filtration are equivalent to the spectra $L(k)_q$. Then we can show the E-homology of the spectrum $L(k)_q$ is isomorphic to to the dual of kth term of the Koszul resolution for Δ^q. In spectral sequence's way, this can be described as

$$Ext^s_{\Delta^q}(\tilde{E}_{n,t}(S^q), \bar{E}_{n,t}) \Rightarrow E_{n,q+t-s}(\Phi_{K(n)}(S^q)) \tag{16}$$

for q odd. The attaching map of Goodwillie tower can be studied simultaneously in this approach. This approach is used by [32] for calculating $E_{n*}(\Phi_{K(n)}(S^{2m+1}))$.

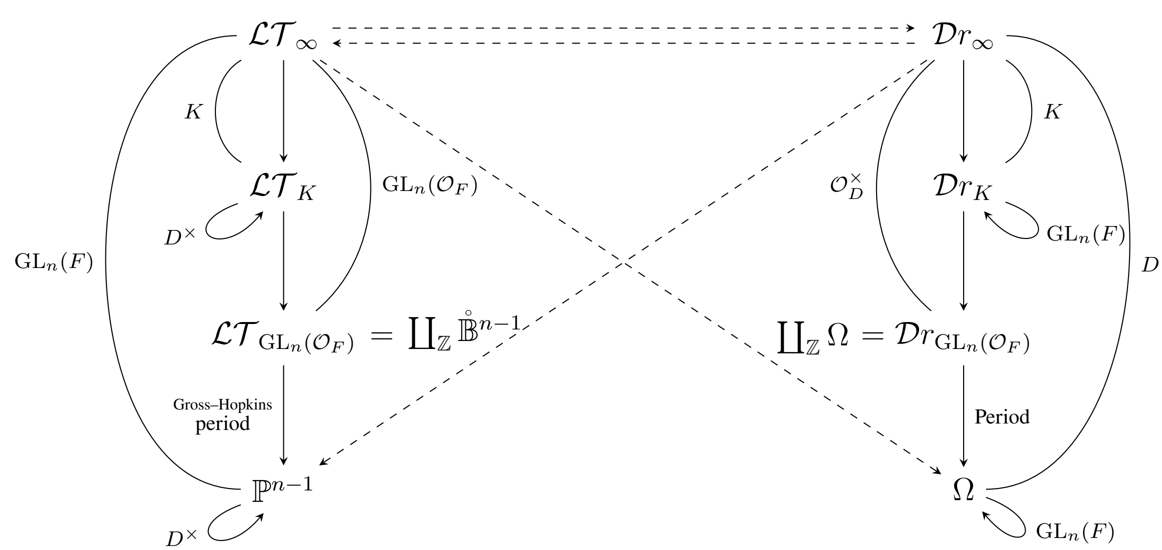
1.6 The problem we are facing and the possible solution

In the above calculation, we found that there are too many potential nontrivial differentials in the Adams-Novikov spectral sequence for most cases. We have few tools to deal with them since $\Phi_{K(n)}(S^{2m+1})$ is not a ring spectrum in general. This makes the calculation along these approaches unavailable for many cases. As a result, we need to develop a new approach to fix this problem. Roughly speaking, the idea is "switch" the order of spectral sequence.

$$Goodwillie \quad tower \xrightarrow{(-)^{GL_n(\mathbb{F}_p)}} H^*_c(G_n, E_{n*}(\Phi_{K(n)}(S^q))) \xrightarrow{(-)^{D^\times}} \pi_*(\Phi_{K(n)}(S^q)). \tag{17}$$

$$Goodwillie \quad tower \xrightarrow{(-)^{D^\times}} \quad ? \quad \xrightarrow{(-)^{GL_n(\mathbb{F}_p)}} \pi_*(\Phi_{K(n)}(S^q)). \tag{18}$$

It is inspried by the following duality in algebra geometry:



The information of each odd sphere is encoded by the Koszul complex, the dimension of these spheres induces a filtration. This information can be seen as a sheaf on $\mathcal{LJ}_{G_n}$ for each Morava stabilizer group. By acting homotopy fixed point spectral sequence on that, we can get the (completed) E_n-homology for $\Phi_n(S^q)$. To understand what happened after this duality is the work I need to do in the further study.

1.7 The structure of this article

This report organizes as follows. We divide the background ingredient into two parts: the chromatic part and the calculation part. These instructions aim to explain the vague details in the above introduction. To be specific, in the chromatic part, we will introduce the Morava K-theory, v_n-self map, Bousfield localization and equivalence, and the Bousfield-Kuhn functor.

For calculation, we will introduce some related spectral sequences such as the generalized Adams spectral sequence, Atiyah Hirzebruch spectral sequence and the Bousfield-Kan spectral sequence[1]. Then, we will briefly introduce some common parts of the known approaches, such as the Goodwillie tower, Steinberg summand and topological Andre-Quillen cohomology.

Finally, we will formulate the problem we are facing and give a work plan in the last section.

2 Background of chromatic homotopy theory

2.1 Morava K-theory

Morava K-theory was first developed in the research of complex oriented bordism theory MU and the formal group laws related to it. If the readers are interested in the history of it, they can refer to [31].

By considering the classifying map $m : \mathbb{C}P_\infty \wedge \mathbb{C}P_\infty \to \mathbb{C}P_\infty$, a formal group law F_{MU} is decided. Its p-local part decided a spectrum BP, which also decided a formal group law F_{BP}.

For a formal group law F, we define $f +_F g = F(f(x), g(x))$, and for any $n > 0$,

$$[n]_F(x) := \underbrace{x +_F \cdots +_F x}_{n} \tag{19}$$

With the above definition, we can give the explicit structure of F_{BP}

Theorem 2.1 (Hazewinkel)**.** *Let p be any prime. There is an isomorphism of $\mathbb{Z}_{(p)}$-algebras*

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots] \tag{20}$$

[1]This spectral sequence relates to the Goodwillie tower and the homotopy fixed point spectral sequence.

where the generators $v_i \in BP_{2(p^i-1)}$ *may be chosen to be the coefficients of* x^{p^i} *in the series*

$$[p]_{F_{BP}}(x) = \sum_{i>0} v_i x^{p^i} \tag{21}$$

The height of a formal group law(of commutative $\mathbb{F}_p$-algebra A) is the power of the leading term of the series of $[p]_F(x)$ logarithmic over p. If $[p]_F(x) = 0$, the height of F is defined to be ∞. Consider the ring homomorphism $\theta_n : BP^* \to A$ defined by $\theta_n(v_n) = 1 and \theta_n(v_i) = 0, for i \neq n$. Put $F_n(x, y) = (\theta)_* F_{BP}$. From the above theorem, we know F_n is height n. By the Landweber exact functor theorem [22], F_n decides a complex-oriented cohomology theory. This cohomology theory is $K(n)$.

For geometry construction, we can get Morava K-theory by killing generators in BP. Some other relevant spectra are also defined here:

$$BP\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots v_n] P(n) \cong \mathbb{F}_p[v_n, v_{n+1}, \cdots] k(n) \cong \mathbb{F}_p[v_n] \tag{22}$$

The spectrum $k(n)$ is the (-1)-connected version of the spectrum $K(n)$ of Morava K-theory. Using $k(n)$ one defines $K(n)$ by

$$K(n) = holim[\Sigma^{-2i(p^n-1)} k(n) \to k(n)] \tag{23}$$

Similiarly, the Morava E-theory $E(n)$ can be defined as the homotopy limit of $BP\langle n\rangle$.

In conclusion, we can summarize the above properties into a theorem:

Theorem 2.2. *Let* p *be any prime. For all integers* $n \geq 1$ *there is a multiplicative,* $2(p^n-1)$*-periodic and complex-oriented cohomology theory* $K(n)^*(-)$ *with coefficient ring*

$$K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}] \tag{24}$$

where v_n *is of degree* $|v_n| = 2(p^n - 1)$ *and whose associated formal group law* $F_n(x, y)$ *satisfies the relation*

$$[p]_{F_n}(x) = v_n x^{p^n}. \tag{25}$$

In addition, If p *is odd, the product on* $K(n)^*(-)$ *is commutative, for* $p = 2$ *it is non-commutative.*

A very important aspect of the Morava K-theories is the fact that they are strongly related to BP-theory and complex cobordism via several types of intermediate spectra. The calculation about $K(n)$ is also useful to the calculation of stable homotopy group of sphere [3] and [26].

Theorem 2.3. *Let* N *be a* BP_*BP*-comodule in which every element is* I_n*-torsion and* v_n *acts bijectively. Then there is a natural isomorphism*

$$Ext^*_{BP_*BP}(BP_*, N) \cong Ext^*_{\Sigma(n)_*}(E(n)_*, E(n)_* \otimes_{BP_*} N). \tag{26}$$

Theorem 2.4. *The natural projection $BP_* \to K(n)_*$. induces an isomorphism*

$$Ext^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext^*_{K(n)_*K(n)}(K(n)_*, K(n)_*). \tag{27}$$

What's more, $K(n)$ has the unique property. So we can define it in an axiom way. This definition is the most common definition of Morava K-theory today.

Theorem 2.5. *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geq 0$ with the following properties. (We follow the standard practice of omitting p from the notation.)*

- $K(0)_*(X) = H_*(X; \mathbb{Q})$ *and* $\overline{K(0)_*}(X) = 0$ *when* $H_*(X)$ *has only torsion summands.*
- $K(1)_*(X)$ *is one of* $p-1$ *isomorphic summands of mod p complex K-theory.*
- $K(0)_*(pt.) = \mathbb{Q}$ *and for* $n \geq 1$, $K(n)_*(pt.) = \mathbb{Z}/(p)[v_n, V_n^{-1}]$ *where the dimension of* v_n *is* $2p^n - 2$. *This ring is a graded field in the sense that every graded module over it is free.* $K(n)_*(X)$ *is a module over* $K(n)_*(pt.)$.
- *There is a Kunneth isomorphism*

$$K(n)_*(X \prod Y) \cong K(n)_*(X) \otimes_{K(n)_*(pt.)} K(n)_*(Y). \tag{28}$$

- *Let X be a p-local finite CW-complex. If* $\overline{K(n)_*}(X)$ *vanishes , then so does* $\overline{K(n-1)_*}(X)$
- *If X is as above then*

$$\overline{K(n)_*}(X) = K(n)_*(pt.) \otimes \overline{H_*}(X; \mathbb{Z}/(p)) \tag{29}$$

  for n sufficiently large. In particular, it is nontrivial if X is simply connected and not contractible.

2.2 v_n-self map

v_n-self map is the self map of space (or spectrum) we are concerned with in this article. Roughly speaking, it is non-nilpotent, graded by the chromatic level's self-map. It can be detected by Morava K-theory.

Definition 2.1. A p-local finite complex X has type n if n is the smallest integer such that $\overline{K(n)_*}(X)$ is nontrivial. In particular, X has type ∞ if it is contractible.

Theorem 2.6. *Let X and Y be p-local finite CW-complexes of type n for n finite.*

- *There is a self-map* $f : \Sigma^{d+i}X \to \Sigma^d X$ *for some* $i \geq 0$ *such that* $K(n)_*(f)$ *is an isomorphism and* $K(m)_*(f)$ *is trivial for* $m > n$. *(Such a map is called* v_n *map which we will discuss later.) When* $n = 0$ *then* $d = 0$*, and when* $n > 0$ *then* d *is a multiple of* $2p^n - 2$.

- *Suppose $h : X \to Y$ is continuous and both of them have already been suspended enough times to be the target of a v_n-map. Let $g : \Sigma^e Y \to Y$ be a self-map as before. Then there are positive integers i and j with $di = ej$ such that the following diagram commutes up to homotopy. (The "uniqueness" of v_n map.)*

$$\begin{array}{ccc} \Sigma^{di} X & \xrightarrow{\Sigma^{di} h} & \Sigma^{di} Y \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle g^j} \\ X & \xrightarrow{h} & Y \end{array}$$

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2.3 Bousfield localization and Bousfield equivalence

Bousfield localization is a special situation of localization over a spectrum E:

Definition 2.2. Let E_* be a generalized homology theory. A space (or a spectrum) Y is E_*-local if whenever a map $f : X_1 \to X_2$ is such that $E_*(f)$ is an isomorphism, the map

$$[X_1, Y] \xleftarrow{f^*} [X_2, Y] \tag{30}$$

is also an isomorphism.

An E_*-localization of a space or spectrum X is a map η from X to an E_*-local space or spectrum X_E (which we will usually denote by $L_E X$) such that $E_*(\eta)$ is an isomorphism.

The property of E_*-local is stable under the inverse limit, fiber sequence and smash product. But it is not stable under the homotopy inverse limit. In [12] and [11], Bousfield proved that for any homology theory E_* and any space or spectrum X, the localization $L_E X$ exists and is functorial in X.

For a ring spectrum E, its localization is simple:

Theorem 2.7. *If E is a ring spectrum, then $E \wedge X$ is E_*-local for any spectrum X.*

Since $K(n)$ is a ring spectrum, we can easily give the definition of $L_{K(n)}$

Next, we can consider when two different spectra E and F induce the same localization functor. This question leads to the Bousfield equivalence:

Definition 2.3. Two spectra E and F are Bousfield equivalent if for each spectrum X, $E \wedge X$ is contractible if and only if $F \wedge X$ is contractible. The Bousfield equivalence class of E is denoted by $\langle E \rangle$.

We will list some definitions and properties of Bousfield equivalence above:

- $\langle E\rangle \geq \langle F\rangle$, if for each spectrum X, the contractibility of $E \wedge X$ implies that of $F \wedge X$.
- $\langle E\rangle \wedge \langle F\rangle = \langle E \wedge F\rangle$ and $\langle E\rangle \vee \langle F\rangle = \langle E \vee F\rangle$.
- A class $\langle E\rangle$ has a complement $\langle E\rangle^c$ if $\langle E\rangle \wedge \langle E\rangle^c = \langle pt.\rangle$ and $\langle E\rangle \vee \langle E\rangle^c = \langle S^0\rangle$. Here the S^0 is the sphere spectrum.
- The operations $\wedge$ and $\vee$ satisfy the obvious distributive laws:

$$(\langle X\rangle\wedge\langle Y\rangle)\vee\langle Z\rangle = (\langle X\rangle\vee\langle Z\rangle)\wedge(\langle Y\rangle\vee\langle Z\rangle)(\langle X\rangle\vee\langle Y\rangle)\wedge\langle Z\rangle = (\langle X\rangle\wedge\langle Z\rangle)\vee(\langle Y\rangle\wedge\langle Z\rangle) \tag{31}$$

- The localization functors L_E and L_F are the same if and only if $\langle E\rangle = \langle F\rangle$. If $\langle E\rangle \leq \langle F\rangle$ then $L_E L_F = L_E$ and there is a natural transformation $L_F \to L_E$.

Bousfield equivalence can explain why we consider the p-component of homotopy groups separately. Let $S^0\mathbb{Q}$ denote the rational sphere spectrum, $S^0_{(p)}$ the p-local sphere spectrum, and $S^0/(p)$ the mod p Moore spectrum. Then we have

- $\langle S^0/(p)\rangle = \langle S^0\mathbb{Q}\rangle \vee \langle S^0/(p)\rangle$
- $\langle S^0\rangle = \langle S^0\mathbb{Q}\rangle \vee \bigvee_p \langle S^0/(p)\rangle$
- $\langle S^0/(p)\rangle \wedge \langle S^0\mathbb{Q}\rangle = \langle pt.\rangle$
- $\langle S^0/(p)\rangle \wedge \langle S^0/(q)\rangle = \langle pt.\rangle$ (The orthogonal property)

For MU, there is a similiar result. Any readers interested in this can refer to Chap 7.3 of [28].

What's more, the Class invariance theorem shows that the type of p-local spectrum decided its class in Bousfield equivalence, which implies that the choice of type n spectrum V and self-map v doesn't infect the result of localization.

Theorem 2.8. *Let X and Y be p-local finite CW-complexes of types m and n respectively. Then $\langle X\rangle = \langle Y\rangle$ if and only if $m = n$, and $\langle X\rangle < \langle Y\rangle$ if and only if $m > n$.*

Bousfield localization of $E(n)$ can be used to give the chromatic filtration of spectrum X.

Definition 2.4. $L_n X$ is $L_{E(n)}X$ and $C_n X$ denotes the fiber of the map $X \to L_n X$

With the following theorem, we can calculate $BP_*(L_n X)$ in terms of $BP_*(X)$

Theorem 2.9 (Localization theorem)**.** *For any spectrum X, $BP \wedge L_n Y = Y \wedge L_n BP$. In particular, if $v_{n-1}^{-1} BP_*(Y) = 0$, then $BP \wedge L_n Y = Y \wedge v_n^{-1} L_n BP$.*

Then we can define the chromatic tower and chromatic filtration of X:

Definition 2.5. The chromatic tower for a p-local spectrum X is the inverse system

$$L_0 X \leftarrow L_1 X \leftarrow L_2 X \leftarrow \cdots X. \tag{32}$$

The chromatic filtration of $\pi_*(X)$ is given by the subgroups

$$ker(\pi_*(X) \to \pi_*(L_n X)) \tag{33}$$

The advantage of this definition is that there are methods of computing $\pi_*(L_n X)$. In particular, suppose X is a p-local finite CW-complex of type n with v_n-self map f. Let $\hat{X}$ be the telescope decided by the v_n-self map. Then $K(n)_*(f)$ is an isomorphism. The same is true of $K(i)_*(f)$ for $i < n$ since $K(i)_*(X) = 0$. Hence $E(n)_*(f)$ is an equivalence. This means that the map $X \to L_n X$ factors uniquely through the telescope $\hat{X}$. That is, $\lambda : \hat{X} \to L_n X$. Moreover, $BP_*(L_n X) = v_n^{-1} BP_*(X)$ and λ is a BP_*-equivalence.

For the v_n-periodic homotopy group, Ravenel gave a conjecture

Theorem 2.10. *Let X be a p-local finite CW-complex of type n. Is the map λ always an equivalence?*

It is true for $n = 0, 1$, but for $n \geq 2$, this conjecture is supposed to be false. However, since we only have enough tools to compute the chromatic part, we usually calculate the $\pi_*(L_{K(n)} X)$ instead.

2.4 Bousfield-Kuhn functor

Now since we are concerned about the unstable part, all of the tools we have is designed to calculate in the stable range. So we need a functor to pull $Space_*$ back to Sp. That is the Bousfield Kuhn functor.

By construction, $\Phi_V(X)$ is a t-periodic spectrum whose 0^{th} space is given by the direct limit of the sequence

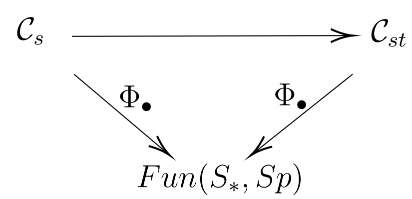
$$Map_*(V, X) \to Map_*(\Sigma^t V, X) \to Map_*(\Sigma^{2t} V, X) \to \cdots \tag{34}$$

In this definition, we need explicit V and v. To get rid of these, we need $\mathcal{C}_t$ to be the ∞-category whose objects are finite pointed spaces V equipped with a v_n-self map $v : \Sigma^t V \to V$.

For each integer $t > 0$, the construction $(V, v) \to \Phi_V$ determines a functor of ∞-categories

$$\Phi_\bullet : \mathcal{C}_t^{op} \to Fun(Space_*, Sp) \tag{35}$$

By sending (V, v) to (V, v^s), we get a functor $\mathcal{C}^t \to \mathcal{C}^{st}$ which fits into the following diagram.



By sending (V, v) to $(\Sigma V, \Sigma v)$, we get a functor $\mathcal{C}^t \to \mathcal{C}^t$ which fits into the following diagram.

$$\begin{array}{ccc} \mathcal{C}_s & \xrightarrow{\qquad\qquad} & \mathcal{C}_{s+1} \\ & \searrow^{\Phi_\bullet} \quad {}^{\Sigma\Phi_\bullet}\swarrow & \\ & Fun(S_*, Sp) & \end{array}$$

Using these observations, we see that the functors $\Phi_\bullet$ can be amalgamated to a single functor $\mathcal{C}' \to Fun(Space_*, Sp)$, where $\mathcal{C}'$ is obtained from the ∞-categories $\mathcal{C}_t$ by taking a direct limit along the transition functors given by suspension and raising self-maps to powers; more precisely, we take $\mathcal{C}'$ to be the direct limit of the sequence

$$\mathcal{C}_{1!} \to \mathcal{C}_{2!} \to \mathcal{C}_{3!} \to \cdots \tag{36}$$

where the map from $\mathcal{C}_{(m-1)!}$ to $\mathcal{C}_{m!}$ is given by $(V, v) \to (\Sigma V, \Sigma(v^m))$. We will abuse notation by denoting this functor also by $\Phi_\bullet : \mathcal{C}' \to Fun(Space_*, Sp)$. We can prove that the ∞-category $\mathcal{C}'$ can be identified with the full subcategory of Sp spanned by the finite spectra of type $\geq n$ which is denoted as $Sp^{fin}_{\geq n}$. Then the functor $\Phi_\bullet$ can be considered as $\Phi_\bullet : Sp^{fin}_{\geq n} \to Fun(Space_*, Sp)$. It can be described informally as follows: if E is a finite spectrum of type n, then we can choose some integer k such that $\Sigma^k E \simeq \Sigma^\infty V$, where V is a finite space of type n which admits a v_n-self map. In this case, we have $\Phi_E = \Sigma^k \circ \Phi_V$.

By consider $\Phi_n(X) = \varprojlim_{E \to S^0} \Phi_E(X)$, we can get the Bousfield-Kuhn functor. This functor is unique and satisfies the following properties.

Proposition 2.11.
- *For every pointed space X, the spectrum $\Phi_n(X)$ is $T(n)$-local.*
- *There are equivalences $\Phi_E(X) \simeq \Phi_n(X)^E$, depending functorially on $E \in Sp^{fin}_{\geq n}$ and $X \in Space_*$.*
- *$\Phi_n(X)$ is left exact.*
- *If X is a spectrum, then $\Phi_n \Omega^\infty X = L_{T(n)} X$ (It shows that the unstable situation is compatible with the stable situation.)*

In particular, it is often more convenient to describe $\Phi_n(X)$ as the homotopy limit $\lim \Phi_{E_K}(X)$, where

$$E_0 \to E_1 \to E_2 \to \cdots \tag{37}$$

is a direct system of type n spectra which is cofinal among all finite type n spectra with a map to S^0. Such a cofinal system can always be found: for example, in the case $n = 1$, we can take the system of Moore spectra

$$\Sigma^{-1} S^0/(p) \to \Sigma^{-1} S^0/(p^2) \to \cdots \tag{38}$$

The v_n-periodic homotopy equivalence $f : X \to Y$ induces a homotopy equivalence of spectra $\Phi_n(X) \to \Phi_n(Y)$. So this conversion from space to spectrum preserves all of the information of the v_n-periodic homotopy group. Actually, we can factor $\Phi_n(X)$ out as follows

$$Space_* \xrightarrow{M_n^f} Space_*^{v_n} \to Sp. \tag{39}$$

where $Space_*^{v_n}$ is the category of pointed spaces which support v_n-self map. The functor $\Phi : Space_*^{v_n} \to Sp$ admits a left adjoint $\Theta : Sp \to Space_*^{v_n}$

The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$. We are more concerned about this part since we have enough tools to calculate it instead of $T(n)$-local spectrum.

3 Background of calculation

3.1 Generalized Adams spectral sequence

Adams spectral sequence is the most important tool for us to calculate the homotopy group. It gives us a method to extract the homotopy information from the mod p cohomology, or other cohomology theory. If we use the ordinary mod p cohomology, we get the classical Adams spectral sequence. If we use the BP, we get the Adams-Novikov spectral sequence. For other cohomology theories, there is some spectral sequence, but the above two spectral sequences are widely used.

In this subsection, unless otherwise stated all homology and cohomology groups will have coefficients in $\mathbb{Z}_p$ for a prime number p. We will start at the classical Adams spectral sequence.

Theorem 3.1 (Adams,[1])**.** *Let X be a spectrum with a finite dimension of $H^*(X)$. There is a spectral sequence*

$$E_*^{*,*}, d_r : E_r^{s,t} \to E_r^{s+r,t+r-1} \tag{40}$$

such that

$$E_2^{s,t} = Ext_{\mathcal{A}_p}^{s,t}(H^*(X), \mathbb{Z}_p) \Rightarrow \pi_*(X) \otimes \mathbb{Z}_{(p)} \tag{41}$$

Here $\mathcal{A}_p$ is the mod p Steenrod algebra.

What's more, Adams spectral sequence is multiplicative if X is a ring spectrum.

Adams spectral is induced by the Adams resolution, which is a tower such that each fiber is the wedge of some copies of Eilenberg-MacLane space (with some possible suspension).

$$\begin{array}{cccccccc} X & \xleftarrow{=} & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} \\ & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & \\ & & K_0 & & K_1 & & K_2 & \end{array}$$

We can roll it into an exact couple and get a spectral sequence. From the algebra perspective, it is a (minimal) resolution of $H^*(X)$ as an $\mathcal{A}_p$-module. That also leads to the computer computation of E_2 page of classical Adams spectral sequence.

If we repeat the same thing for other cohomology theories, we get the generalized Adams spectral sequence. For a given cohomology theory E_* (with some mild restrictions, such as $E_*(E)$ has a E-comodule structure), we have

Definition 3.1. An E_*-Adams resolution for X is a diagram

$$\begin{array}{cccccccc} X & \xleftarrow{=} & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} \\ & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & \\ & & K_0 & & K_1 & & K_2 & \end{array}$$

such that for all $s \geq 0$ the following conditions hold

- X_{s+1} is the fiber of f_s.
- $E \wedge X_s$ is a retract of $E \wedge K_s$, i.e., there is a map $h_s : E \wedge K_s \to E \wedge X_s$ such that $h_s(E \wedge f_s)$ is an identity map of $E \wedge X_s$. In particular, $E_*(f_s)$ is a monomorphism.
- K_s is a retract of $E \wedge K_s$.
- $Ext^{t,u}(E_*(K_s)) = \pi_u(K_s)$ for $t = 0$ and it equals to 0 otherwise.

Here the explicit definition of Ext and the restrictions of E_* at here can refer to the Appendix A1 and Chapter 2.2 of [27].

Like the classical Adams spectral sequence, this spectral sequence can detect the E-component of $\pi_(X)$. That is,

Definition 3.2. An E-completion $\hat{X}$ of X is a spectrum such that

- There is a map $X \to \hat{X}$ inducing an isomorphism in E_*-homology.
- $\hat{X}$ has an E_*-Adams resolution $\{\hat{X}_s\}$ with $\lim \hat{X}_s = pt$.

The above resolution induces a spectral sequence:

Theorem 3.2. *An E_*-Adams resolution for X leads to a natural spectral sequence $E_*^{*,*}$ with $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that $E_2^{s,t} = Ext(E_*(X)) \Rightarrow \pi_*(\hat{X})$*

If we let $E = BP$, we get the Adams-Novikov spectral sequence.

Finding an analogue of the minimal resolution of the comodule is difficult. So we use cobar construction to get the canonical E_*-Adams resolution. It is useful for proof, but for calculation, the cobar complex is too big to compute. Some other tools are developed to solve this problem. Such as the May spectral sequence and lambda algebra. However, it's not relative to this article. Readers interested in this topic can read Chapter 3 of [27]. And the details of calculating the Adams-Novikov spectral sequence can be found in Chapter 4.

3.2 Atiyah-Hirzebruch spectral sequence

The Atiyah-Hirzebruch spectral sequence is a generalization of the Serre spectral sequence, which can help us calculate the generalized cohomology. For a (homotopy) fiber sequence $F \to X \to Y$, the corresponding E-Atiyah-Hirzebruch spectral sequence is

$$E_2^{s,t} = H^s(X, E^t(X)) \Rightarrow E^*(X). \tag{42}$$

Here the H is the ordinary cohomology.

Analog to Adams spectral sequence, the Atiyah-Hirzebruch spectral sequence is multiplicative if the cohomology theory E is decided by a ring spectrum. The Kronecker pairing $\langle -, - \rangle : E^*(X) \otimes E_*(X) \to \pi_*(X)$ passes to a page-wise pairing of the corresponding Atiyah-Hirzebruch spectral sequence[21]

$$\langle -, - \rangle_r : \varepsilon_r^{n,-s} \otimes \varepsilon_{n,t}^r \to \pi_{s+t}(X). \tag{43}$$

3.3 Bousfield-Kan spectral sequence

We'll work in $sSet$, so that everything is connective. Consider a tower of fibrations:

$$\cdots Y_s \xrightarrow{p_s} Y_{s-1} \xrightarrow{p_{s-1}} Y_{s-2} \cdots \tag{44}$$

for $s \geq 0$ and $Y := \lim_{\leftarrow} Y_s$, and F_s is the fiber of p_s.

By acting π_* on it and rolling it into a spectral sequence, we have:

Theorem 3.3. *In this situation, there is a spectral sequence, called the Bousfield-Kan spectral sequence:*

$$E_1^{s,t} = \pi_{t-s} F_s \Rightarrow \pi_{t-s} Y \tag{45}$$

This spectral is useful if we apply it on the Tot tower. In [15], the author shows how to use this method to get the homotopy fixed point spectral sequence.

Definition 3.3. Let $X^\bullet$ be a cosimplicial object in $sSet$. Then, its totalization is the complex

$$Tot(X^\bullet) = sSet(\Delta^\bullet, X^\bullet) \tag{46}$$

and

$$Tot_n(X^\bullet) = sSet(sk_n\Delta^\bullet, X^\bullet) \tag{47}$$

$Tot_n(X^\bullet) \to Tot_{n-1}(X^\bullet)$ is a fiber in Reedy model structure. If $C, D \in \mathcal{C}$ and $X^\bullet \to C$ is a simplicial resolution in a simplicial category $\mathcal{C}$, then $Hom(X^\bullet, D)$ is a cosimplicial object, and this spectral sequence can be used to compute homotopically meaningful information about $sSet(C, D)$.

Let G be a group, and X be a spectrum with a G-action. Then, the homotopy fixed points of X are

$$X^{hG} := F((EG)_+, X)^G \tag{48}$$

i.e. the G-equivariant maps $(EG)_+ \to X$. The bar construction gives us a simplicial resolution of $(EG)_+$, producing a cosimplicial object that can be plugged into the Bousfield-Kan spectral sequence. Specifically, we write $EG = B^\bullet(G, G, *)$, add a disjoint basepoint, and then take maps into X.

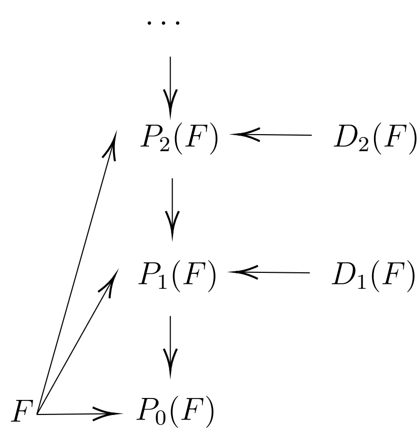
Theorem 3.4. *If X is a spectrum with a G-action, there is a spectral sequence, called the homotopy fixed-point spectral sequence, with signature*

$$E_2^{s,t} = H^s(G, \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG}) \tag{49}$$

3.4 The Goodwillie tower

In general, the Goodwillie tower describe the homogeneous degree d part for each $d \geq 0$ of a functor $F : \mathcal{C} \to \mathcal{D}$.

Theorem 3.5 (Goodwillie,[18])**.** *Given a homotopy functor $F : \mathcal{C} \to \mathcal{D}$ there existsa natural tower of fibrations under $F(X)$*



such that

- *P_dF is d-excisive.*
- *$e_d : F \to P_dF$ is the universal weak natural transformation to a d-excisive functor*

However, since the Bousfield-Kuhn functor has the commutative property with Goodwillie tower as we introduce in the introduction, in this article we only concern about the Goodwillie tower of Id. The theorem of Goodwillie can be rewritten as follows.

Theorem 3.6 (Goodwillie)**.** *Let X be a simply connected pointed space. Then X can be realized as the homotopy limit of a tower*

$$\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X) \tag{50}$$

with the following features:

- *The map $X \to P_1(X)$ agrees with the unit map $X \to \Omega^\infty \Sigma^\infty X$.*
- *Each of the homotopy fibers $D_nX = fib(P_n(X)\text{ß}P_{n-1}(X))$ is an infinite loop space $\Omega^\infty((\Sigma^\infty X)^n \wedge \mathcal{O}(n))_{h\Sigma_n}$, where $\mathcal{O}(n)$ can be explicit defined.*
- *$\mathcal{O}(t)$ is the $\partial_t(F)$, which can be defined as*

  $$\Omega^\infty \partial_n(F) \simeq co \lim_{k_1,\cdots,k_n} \Omega^{k_1+\cdots+k_n} cr_n(F)(S^{k_1},\cdots,S^{k_n}) \tag{51}$$

  Here the cr_n is a functor $cr_n : Space_*^n \to Space_*$ defined by a formula

  $$cr_n(F)(X_1,\cdots,X_n) = tfib\left(S \to \left(\bigvee_{i \notin S \subseteq [n]} X_i\right)\right) \tag{52}$$

  The $tfib$ is the total fiber, which is defined as

  $$tfib(X) := fib\left(\mathcal{X}(\emptyset) \to \lim_{\emptyset \neq S \in P(I)} \mathcal{X}(S)\right) \tag{53}$$

  The $\mathcal{X}$ is the I-cube, which is a functor from $P(I) \to Space_*$ and $P(I)$ is the set of subsets of a finite set I.

If we apply the Bousfield-Kuhn functor on the Goodwillie tower, we have

$$\cdots \to \Phi_n P_4(X) \to \Phi_n P_3(X) \to \Phi_n P_2(X) \to \Phi_n P_1(X) \simeq L_{T(n)}\Sigma^\infty X \tag{54}$$

and the homotopy fiber $D_n(X)$ turns to:

$$\Phi_n D_k X = L_{T(n)}((\Sigma^\infty X)^k \wedge \mathcal{O}(t))_{h\Sigma_k} \tag{55}$$

In general, the homotopy limit of this tower need not be $\Phi_n(X)$ since the functor Φ_n does not commute with infinite homotopy limits. But for sphere, this is true. When X is a sphere, the tower stabilizes ($\Phi_n D_k X \simeq 0$, when $k \gg 0$.)

Come back to the Goodwillie tower of Id. In computational use, the above tower gives us a Bousfield-Kan spectral sequence:

$$E_2^{s,t} : \pi_s((\Sigma^\infty X)^t \wedge \mathcal{O}(t))_{h\Sigma_t} \Rightarrow \pi_s X. \tag{56}$$

The $\mathcal{O}(t)$ is

$$\partial_n(Id) \simeq D(\Sigma\infty\Delta_n) \tag{57}$$

The explicit definition of Δ_n can be found in [1]

In particular, when X is S^K, the sphere of dimension k, we have the following spectral sequence:

$$E_2^{s,t} : \pi_s(\Sigma^{tk}\mathcal{O}(t))_{h\Sigma_t} \Rightarrow \pi_s(S^k). \tag{58}$$

For more details about the general Bousfield-Kan spectral sequence the readers can refer to [5] or Lurie's lecture notes.

In [32], Zhu calculate the completed E-homology of $\Phi_2(S^{2m+1})$. Then we can use this and the homotopy fixed point spectral sequence to calculate the target homotopy group. That is,

$$E_2^{s,t} = H^s(G_n, \hat{E}_*(\Phi_2(S^{2m+1}))). \tag{59}$$

3.5 Steinberg summand

The Steinberg idempotent is the element $e = |GL_n(\mathbb{F}_P)/U_n|\Sigma_{w\in W}\epsilon n_w\mathcal{B}$ in the group ring $\mathbb{Z}_{(p)}[GL_n(\mathbb{F}_P)]$. Here W is the Weyl group, which is the permutation group in n elements in the case of GL_n, $\epsilon(w)$ is the parity of the permutation, n_w its representative in the normalizer of the split maximum torus, and $\mathcal{B}$ is the sum of elements in the Borel subgroup, i.e. the upper triangular matrices, and U is the unipotent subgroup in the Borel subgroup, i.e. those matrices with diagonals 1. This element generates a projective irreducible representation of $GL_n(\mathbb{F}_P)$. Since e is an idempotent, for any $GL_n(\mathbb{F}_P)$-module we can define its Steinberg summand to be

[1] https://www.math.ias.edu/ lurie/ThursdayFall2017/Lecture11-Derivatives.pdf

the image of e. This also extends to any spectrum acted by $GL_n(\mathbb{F}_P)$. Concretely, the coefficients in e are in $\mathbb{Z}_{(p)}$, so when localized at p, any such spectrum has a self map defined by e, and its Steinberg summand is just the fiber of $1-e$. Let $B\mathbb{F}_P^n$ be the classifying space of the additive group of the vector space over $\mathbb{F}_P$ of dimension n. Then $GL_n(\mathbb{F}_P)$ acts on it. Let $\bar{\rho}$ be the reduced regular representation of $B\mathbb{F}_P^n$, i.e. the sum of all the nontrivial irreducible representations. This representation is $GL_n(\mathbb{F}_P)$-equivariant, and also its multiples $k\bar{\rho}$. So the vector space defined by them are also acted by $GL_n(\mathbb{F}_P)$. By taking the Thom spectra, we construct spectra $(B\mathbb{F}_P^n)^{k\bar{\rho}}$ with $GL_n(\mathbb{F}_P)$-action. Denote by $L(n)_k$ the Steinberg summand of $(B\mathbb{F}_P^n)^{k\bar{\rho}}$. Then it is shown in [2] that the Goodwillie derivatives of spheres $D_{p^N}S^k$ are homotopy equivalent to $\Omega^\infty\Sigma^{k-n}L(n)_k$. We will abbreviate $L(n)_1$ by $L(n)$. There is yet another description of the $L(n)$'s. Let $SP^n(S)$ be the nth symmetric power of the sphere spectrum. Then by Dold-Thom theorem, $SP^\infty(S)$ is a model for $H\mathbb{Z}$. There is the filtration $SP^1(S) \to SP^p(S) \to SP^{p^2}(S) \to \cdots$. One finds $L(n) = \Sigma^{-n}SP^{p^n}(S)/SP^{p^{n-1}}(S)$.

3.6 Topological Andre-Quillen (co)homology

Suppose that R is a commutative S-algebra, and that A is an augmented commutative R-algebra. Topological Andre-Quillen homology of A (relative to R) was defined by Basterra in [4] as a suitably derived version of the cofiber of the multiplication map on the augmentation ideal:

$$TAQ^R(A) = I(A)/I(A)^{\wedge 2} \tag{60}$$

If M is an R-module, then topological Andre-Quillen homology and cohomology of A with coefficients in M are defined respectively as

$$TAQ^R(A;M) = TAQ^R(A) \wedge_R M \tag{61}$$

$$TAQ_R(A;M) = F_R(TAQ^R(A), M) \tag{62}$$

As with TAQ^R, we define $TAQ_R(A) := TAQ_R(A;R)$

Basterra shows in above article that TAQ admits a simplicial presentation using the monadic bar construction. This connects TAQ with the Koszul complex. The details can be found in [7].

4 Formulation of problem and the work plan.

Finally, we will formulate the question we want to solve and the method we want to try.

Since in the known approach, we have no tools to deal with thoes differentials in last spectral sequence, we are trying to swap the order of these two spectral sequence in the known method.

This is motivated by a duality between Lubin-Tate tower and Drinfeld tower in algebraic geometry.

To finish this duality, we need to understand those concepts appear in the known methods. This part is supposed to be finished in summer vacation. Then, we will try to find their dual and bulid a new frame of calculation. If we achieve this target, we will try some explicit calculation by the new method as an example.

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