Out line of this lecture

Part 0: Why we need Adams Spectral Sequence? (as Adams SS)

① Postnikov tower + Hurewicz thm + Serre SS

Advantage: direct

Disadvantage: Hard (even impossible) to calculate

② Whitehead tower

Advantage: could be calculated

Disadvantage: Calculation of it is complex and make no full use of the cohomology structure (Steenrod operator)

③ Adams SS

Advantage (Main idea): Killing cohomology as $\mathcal{A}_p$-mod instead of $\mathbb{Z}$-mod (or $\mathbb{Z}_p$-mod)

Disadvantage: Not so explicite related to homotopy group

Part 1 Setting up of the Adams SS

① Adams resolution, its geometry realization, and why Adams SS is a SS.

② Pf of the Adams thm

1) It is a spectral sequence, and the E_r page of it (especially E_2 page)

2) It converge to sth relative to the $\pi_*^s(S^n)$

Outline of this lecture (next lecture)

⋮

③ Mutliplication struture of Adams SS

Part 2 Example of explicit calculation of Adams SS's E_2 page.

① Minimal resolution

1) Additive structure

2) Mutliplictive structure

3) Idea about computer calculation and diffculity about it.

② Other methods

1) May SS

2) Lambda algebra

only briefly introduction of idea.

Part 3: Deeper structure on Adams SS and differential over it.

1) Massey product and Toda bracket

2) Simple example of "guess" the differential.

① Elementwise

② Element-free method (If the calculation is meanful at small range ⋯)

Appendix: How to read the calculation result of the Adams SS?

All of the screenshot in this note is from "Homotupicial Topology", Anatoly Fumenko.

Part 0. The history of cal. $\pi_*(S^n)$

Note: We have such tools to compute π_*: Hurewicz thm and long exact sequence of htpy gp.

① Postnikov tower / Postnikov Approximation.

Every connected CW complex can be approximated by a twisted product of Eilenberg–MacLane space:

$$\begin{tikzcd}
Y_1 \arrow[d, equal] & Y_2 \arrow[l, "K(\pi_2,2)"'] & \cdots \arrow[l, "K(\pi_3,3)"'] & Y_n \arrow[l] & \cdots \arrow[l] \\
K(\pi_1,1) & & & X \arrow[u] \arrow[ulll, bend left] \arrow[lllu, bend right] &
\end{tikzcd}$$

, $X \to Y_q$ induce an iso. in π_i, $i \leq q$.

Y_q : X, but attaching cells to kill all of the π_{q+k}, $k \geq 1$

Consider the exact htpy seq, we can find fiber of $Y_q \to Y_{q-1}$ is the Eilenberg MacLane space.

e.g. $K(\pi_4, 4) \to Y_4$ $\qquad Y_q = S^3 \cup e^{q+2} \cup \cdots$, $H_q(Y_q) = H_{q+1}(Y_{q+1}) = 0$

$$\begin{array}{c} Y_4 \\ \downarrow \\ K(\mathbb{Z},3) \end{array} \quad \left(\begin{array}{l} H_3 = \mathbb{Z} \\ H_5 = \mathbb{Z}_2 \end{array} \right) \qquad \Rightarrow \pi_q(S^3) = H_{q+1}(Y_{q-1})$$

Problem: Generally we have no idea about $H_*(Y_q)$, since we can't get information about the connection about Y_q, and get $H_*(Y_{q+1})$ from $H_*(Y_q)$ is difficult.

启示：在消灭的过程中，可以逐步的消灭 π_n 中的元素而保持上面的同伦群不变

Idea: we can kill π_n step by step, so we can get information about success. space.

② Whitehead tower. —— generalizes the universal covering of a space.

Idea: just kill successively the htpy gp. above a given dim. + all below dim

$\pi_q \triangleq \pi_q(X)$, $Y = X$, but kill π_q for $q \geq 2$.

$Y = X \cup e^3 \cup \cdots$ is a $K(\pi_1, 1)$ containing X as a subspace.

Ω_*^X : all paths in Y from a base point $*$ to X. $\Omega_*^X \to X$ is a fibration, with fiber $\Omega Y = \Omega K(\pi_1, 1) = K(\pi_1, 0)$

$$\begin{array}{c} K(\pi_1, 0) \to \Omega_*^X \\ \qquad\quad \downarrow \\ \qquad\quad X \end{array}$$

, $X_1 = \Omega_*^X$ is the universal covering of X up to htpy.

Extend it to a tower, we have a tower.

$$\begin{array}{lcl} K(\pi_n, n-1) \to & X_n & \\ & \downarrow & \\ & X_{n-1} & \\ & \downarrow & \\ & \vdots & \\ & \downarrow & \\ K(\pi_1, 0) \to & X_1 & \\ & \downarrow & \\ & X & \end{array}$$

s.t.
① X_n is n-connected
② $\pi_m(X_n) = \pi_m(X)$, if $m > n$
③. The fib. of $X_n \to X_{n-1}$ is $K(\pi_n, n-1)$

(Note, $\pi_{q+1}(X) = H_{q+1}(X_q)$)

Advantage: We know more info about X_n (need info from X_{n-1})

X_{n-1} known, X_n?

- Kill $\pi_q(X_{n-1})$ for $q \geq n+1$, we get a $K(\pi_n, n)$ with a subspace X_{n-1}.
- $X_n = \Omega_*^{X_{n-1}}$ be the space of all paths in $K(\pi_n, n)$
- $X_n \to X_{n-1} = \Omega K(\pi_n, n) = K(\pi_n, n-1)$

analog of the previous steps.

Easy to check $\pi_q(X_n) = \pi_q(X_{n-1})$, $q \geq n+1$; $\pi_q(X_n) = 0$, $q \leq n-2$.

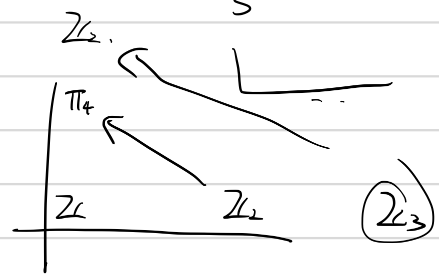
$$0 \to \pi_n(X_n) \to \underset{\pi_n(K(\pi_n, n-1))}{\underset{\parallel}{\pi_n(X_{n-1})}} \xrightarrow{\partial} \pi_{n-1}(\Omega K(\pi_n, n)) \to \pi_{n-1}(X_n) \to 0 \quad \text{exact.}$$

$\pi_n(X_{n-1}) = \pi_n$, and ∂ is an isomorphism (这里省去细节,大约是用 $\pi_n(K(\pi_n, n))$ 中转一下) (Details are omitted)
(Induction)

Application: Serre's Thm: $\pi_*(S^n)$ is torsion, expect dim n or $2n-1$
$\leftarrow n$ odd
$\leftarrow n$ even

$\pi_5(S^3)$: $K(\pi_4, 3) \to X_4$ $\downarrow$ $K(\mathbb{Z}, 2) \to X_3$ $\downarrow$ S^3

	0	1	2	3	4	5	6	7	8	9
$H^* X_3 =$	$\mathbb{Z}$					$\mathbb{Z}_2$		$\mathbb{Z}_3$		$\mathbb{Z}_4$
$H_* X_3$	$\mathbb{Z}$				$\mathbb{Z}_2$		$\mathbb{Z}_3$		$\mathbb{Z}_4$	



$\because$ X_4 4-connected $\Rightarrow \pi_4 = \mathbb{Z}_2$.

And $H_4(K(\mathbb{Z}_2, 3)) = 0$ and $H_5(\cdots) = \mathbb{Z}_2$. , $\mathbb{Z}_3 \to \mathbb{Z}_2$ trivial $\Rightarrow d_6$ trivial.

$$H_5(X_4) = \mathbb{Z}_2 = \pi_5.$$

Idea: 可以发现, 我们求π_i的过程 = 在X_i的(上)同调群中杀掉部分元素的过程,

若在X_i中的非平凡的项少, 可以更方便的得到后面的X_{i+1}

$\Rightarrow$ 有没有更高效的, 在保证X_i可相继计算出来的情况下消灭X_i中元素的手段?

(而且, 有没有办法/限制可以让我们更好的刻画$H^*(K(G,n))$?)

Idea: Find $\pi_i \approx$ killing (co)homology of X_i, so we hope there are less non-trivial obj in X_i

And is there any way to get X_{i+1} and kill elements in X_i efficiently? (keep X_{i+1} computable)

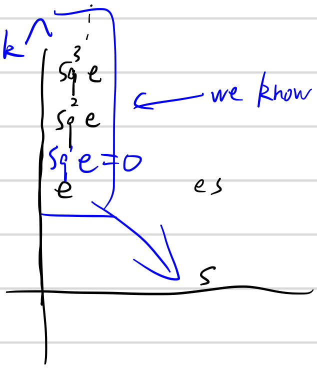
(What's more, a better way to describe $H^*(K(G,n))$?)

Answer) In stable range + mod-p cohomology

e.g.

$$\begin{tikzcd} K(\mathbb{Z}_2, n-1) \arrow[r] & X_1 \arrow[d] \\ & S^n \end{tikzcd}$$

: (n large enough)



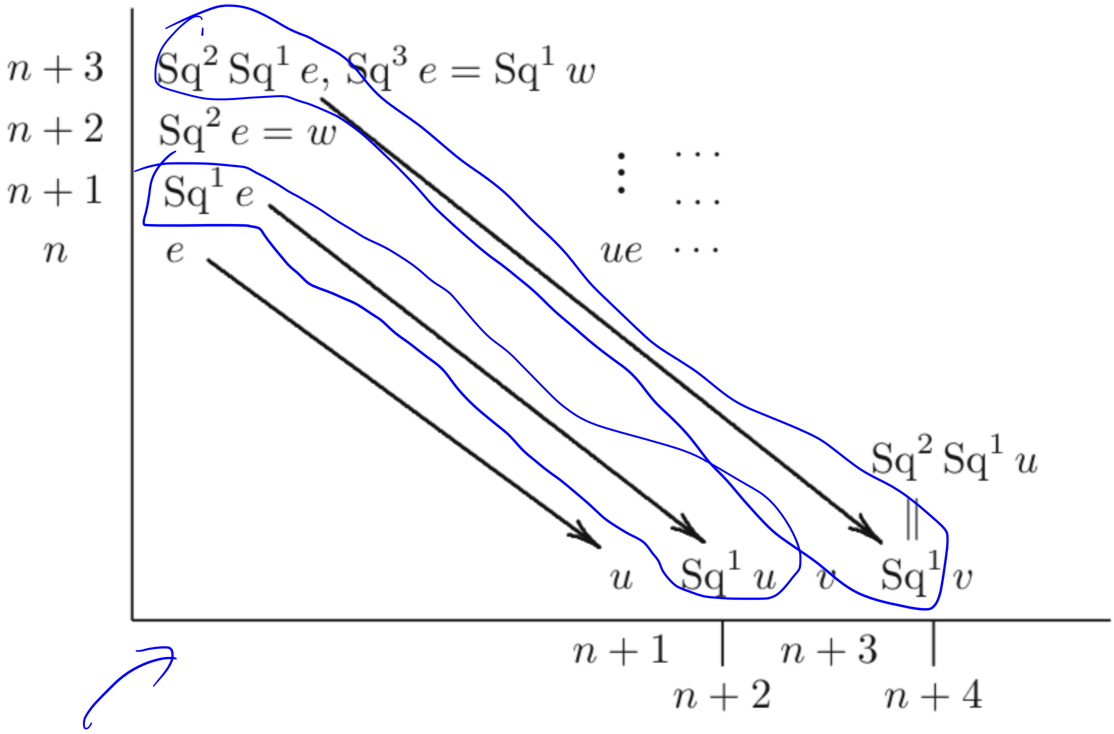
← we know enough info about this part (for k<n)

$Sq^1 e = 0$ (since we have nothing can kill it)

$Sq^2 e = u$, $\quad Sq^1 Sq^2 e = Sq^3 e = Sq^1 u$

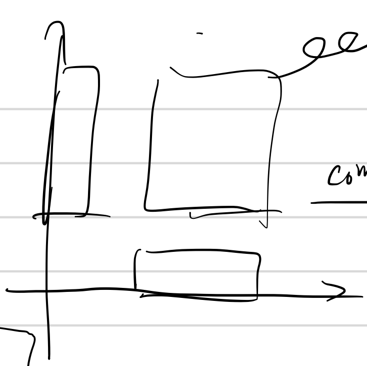
$Sq^2 Sq^2 = Sq^3 Sq^1 e = 0$. So H^{n+3} gen by $Sq^3 e = v$, ...

At stable range, we can easily get the info about $H^*(K(\pi, n))$, and X_i!

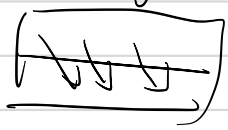


$$\begin{tikzcd} K(\mathbb{Z}_2, n) \arrow[r] & X_2 \arrow[d] \\ & X_1 \end{tikzcd}$$

e.g. $K(\mathbb{Z}_2, n-1) \to X_1$
$\downarrow$
X



compress the stable range as:



e	$Sq^1 e$	$Sq^2 e$	$Sq^3 e$ / $Sq^2 Sq^1 e$
	α	$Sq^1 \alpha$	$\cdots$
$N-1$	N	$N+1$	

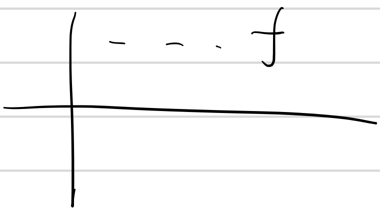
We know about the cohomology of $X(n)$, but we don't know how the cohomology operator acts on $X(n)$!

e.g. Suppose we have $Sq^{30}\alpha = 0$. Then we know $Sq^{30} e$ survive. Called it "f".

And since $Sq^{27} Sq^{30} = 0$.

$Sq^{27} f$ can't exist on upper row. but there could be a $Sq^{27} f = y \in H^*(X_1; \mathbb{Z}_2)$ not 0.

Because f comes from $Sq^{30} e \in H^{N+29}(K(\mathbb{Z}_2, n-1); \mathbb{Z}_2)$, y comes from an element of $H^{N+56}(X; \mathbb{Z}_2)$



$y = Sq^{20} f$

Part 1 Setting up to the Adams SS

Idea: Want to find the p-component from dim N to dim N+n.

① Kill all the cohomology of X mod p in dim $N \sim N+n$.

$$\begin{array}{c} \prod_i K(\mathbb{Z}_p, N+q_i-1) \longrightarrow X(1) \\ \qquad\qquad\qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad\qquad\quad X \end{array}$$

, each i correspondence to an element in $H^{N+q}(X;\mathbb{Z}_p)$

$H^*(\prod_i K(\mathbb{Z}_p, --), \mathbb{Z}_p)$ is a free A_p-mod. (At least in dim $<< N$)

It's an epimorphism $\Rightarrow$ ker of differential could survive to ∞, but nothing survive in bottom row.

Actually, for e, kill $\{Sq^{i_s} e\}$, not kill them separately

Or in algebraically, we kill e as a generator of A_p-mod.

Then we get a free resolution. $\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to H^*(X, \mathbb{Z}_p) \to 0$

corr. to each $Y_i := \prod K(---)$

But how is Adams killing related to the htpy gp?

The elements killed by Adams killing give us an upper estimate for the p-components.

And we need to remove the extra elements (by differential)

But how?

$$\begin{tikzcd} \vdots \arrow[d] & \\ X(2) \arrow[r] \arrow[d] & Y_2 \\ X(1) \arrow[r] \arrow[d] & Y_1 \\ X \arrow[r] & Y_0 \end{tikzcd} \quad \xrightarrow{\pi_*} \quad \begin{tikzcd} \vdots \arrow[d] & \\ \pi_*(X(2)) \arrow[r] \arrow[d] & \pi_* Y_2 \\ \pi_*(X(1)) \arrow[r] \arrow[d] & \pi_* Y_1 \arrow[ul, dashed] \\ \pi_* X \arrow[r] & \pi_* Y_0 \arrow[ul, dashed] \end{tikzcd}$$

$\leftarrow$ connecting homomorphism in long exact seq. of htpy gp.

↙ Geometric realization of Adams resolution.

X be a top. space, Find $\pi_q^s(X) = \pi_{N+q}(\Sigma^N X)$, $N \gg q$.

Fix a prime number p. $\tilde{H}^*(X)$ is an $\mathbb{A}$-mod ($= \mathbb{A}_p$). it have a free resolution as a $\mathbb{A}$-mod
$\tilde{H}^*(X) \leftarrow B_0 \leftarrow B_1 \leftarrow \cdots$

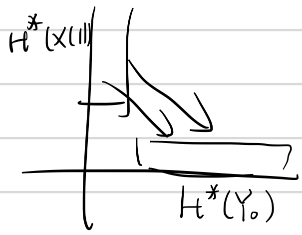
For N large enough, The $\mathbb{A}$-mod $\tilde{H}^*(\Sigma^N X)$ is same as $\tilde{H}^*(X)$, only the grading shifted by N.

$\Sigma^N X \to K(\mathbb{Z}_p, N+q_i)$ corresponding to the classes $\Sigma^N \alpha_i \in H^{N+q_i}(\Sigma^N X)$

All together, we have $f: \Sigma^N X \to Y_0 = \prod_i K(\mathbb{Z}_p, N+q_i)$

$H^*(Y_0)$ in dim $N \sim 2N$ = B_0-mod in dim $0 \sim N$.

And this map induce a homomorphism $\tilde{H}^*(Y_0) \to \tilde{H}^*(\Sigma^N X)$



Denote f as a fibration, fiber $\triangleq X(1)$. Up to dim $2N-3$. $\tilde{H}^*(X(1)) = \mathrm{Ker}[\tilde{H}^*(Y_0) \to H^*(\Sigma^N X))$
(dim -1)

As a result. $\tilde{H}^*(X(1))$ in dim $N-1 \sim 2N-3 \cong \mathrm{Ker}[B_0 \to \tilde{H}^*(X)]$

Do the same thing to $X(1)$, we get $X(2)$, $X(3) \cdots$

Fix a dim $n \ll N$. $\tilde{H}^*(Y_i)$ up to $(N-i+1)+n$ = B_i up to n.

And by extend the fiber sequence inversely, $X(i+1) \to X(i)$ also could be seen as fibration.

$Z_i = Y_i$. But dim are lower by 1. ($Z_i = \Omega Y_i$)

(w Z_i?
$$\begin{tikzcd} Z_i \arrow[d] & \\ X(i+1) \arrow[r] \arrow[d] & X_i \\ Y_{i+1} & \end{tikzcd}$$
$\Rightarrow$ we have $Z_i \to Y_{i+1}$, induce $B_{i+1} \to B_i$)

Use telescope construction $\longrightarrow$ $X(i+1) \hookrightarrow X(i)$, inclusion.

We need a geometric realization of the free resolution.
$\tilde{H}^*(X) \leftarrow B_0 \leftarrow \cdots$
$\tilde{H}^*(X_i) = \mathrm{Ker}(B_{i-1} \to B_{i-2})$
$\tilde{H}^*(Y_i) \sim B_i$

Note: In the above note, we may find that keeping our discussion in stable range by N is a bit complex. That's why we need spectrum

34.2 Setting up of the Adams SS.

There is an improtant prop about π_* in stable range: $\pi_q(A,B) = \pi_q(A/B)$
(which allows us constr. SS)

Consider $(X(s), X(s+r)) \hookrightarrow (X(s+1-r), X(s+1))$

① $E_r^{s,t} \triangleq \mathrm{Im}\left[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))\right]$
(N large enough)

② $E_r^{s,t} \triangleq G_r^{s,t}/D_r^{s,t}$. $G_r^{s,t} = \mathrm{Im}\left[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s), X(s+1))\right]$

original def of Adams.

$D_r^{s,t} = \mathrm{Im}\left[\pi_{N+t-s}(X(s+1-r), X(s)) \to \pi_{N+t-s}(X(s), X(s+1))\right]$

Consider two triples: $(X(s), X(s+1), X(s+r))$

$\underline{(X(s+1-r), X(s), X(s+1))}$

This comes from $(X(s+1-r), X(s), X(s+r))$

$$\begin{tikzcd}
\pi_q(X(s), X(s+r)) \arrow[dr, "g"] & & \\
\pi_{q+1}(X(s+1-r), X(s)) \arrow[u, "f"] \arrow[r, "h"] & \pi_q(X(s), X(s+1)) \arrow[r, "k"] & \pi_q(X(s+1-r), X(s+1))
\end{tikzcd}$$

exact

$h = g\circ f$. $D_r^{s,t} \subset G_r^{s,t}$, $E_r^{s,t} = \mathrm{Im}(k\circ g) = \mathrm{Im}(g)/(\mathrm{Ker}(k) \cap \mathrm{Im}(g))$

$= \mathrm{Im}(g)/(\mathrm{Im}(h) \cap \mathrm{Im}(g)) = G/D.$ ✓

Advantage of the first def: It is stable

Differential:

$$\begin{tikzcd}
\pi_{N+t-s}(X(s), X(s+r)) \arrow[d, "f"] \arrow[dr, "\partial_1"] & \\
\pi_{N+t-s}(X(s+1-r), X(s+1)) \arrow[dr, "\partial_2"] & \pi_{N+(t+r-1)-(s+r)}(X(s+r), X(s+2r)) \arrow[d, "g"] \\
& \pi_{N+(t+r-1)-(s+r)}(X(s+1), X(s+r+1))
\end{tikzcd}$$

$d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,t+r-1}$

by inclusion.

34.3

Thm: Adams Thm: p prime. $\exists$ spectral sequence $\{E_r^{s,t},\ d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,\,t+r-1}\}$ with following prop:

1) $E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\tilde{H}^*(X), \mathbb{Z}_p)$ ($\mathbb{Z}_p$ as a trivial $\mathcal{A}$-mod).

2) $E_{r+1}^{s,t} = \mathrm{Ker}\, d_r^{s,t} / \mathrm{Im}\, d_r^{s-r,\,t-r+1}$

3) For $r>s$. $\mathrm{Im}\, d_r^{s-r,\,t-r+1} = 0$. $E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}$

$\Longrightarrow$ For $t>s$. $\exists$ a chain of subgroups $B^{s+1,\,t+1} \subset \cdots \subset B^{0,\,t-s} \subset \pi_{t-s}^{s}(X)$

4) $\bigcap_{t-s=m} B^{s,t} = \underline{\text{non-}p\text{ component}}$ of $\pi_m^s(X)$ (so by quotient successively, we got the p component)

$\hookrightarrow$ order $\begin{cases} \text{finite} \\ \text{pt order.} \end{cases}$

Pf of 1) and 2)

1. By setting and prop of stable dim,

$E_1^{s,t} = \pi_{N+t-s}(X(s), X(s+1)) = \pi_{N+t-s}(\pi K(\cdots))$, And in the stable dim, $H^*(X(s), X(s+1))$

$\cong \pi^*(\pi K(\cdots)) \xleftrightarrow[\text{generators}]{} \underline{H^*(\pi K(\cdots))}$

Since K_m Eilenberg MacLane, The free generator of $H^*(X(s), X(s+1)) \xleftrightarrow{\text{bijection}} \oplus_t E_1^{s,t}$

(Actually isomorphism with grading shift $N-s$)

And we know that $H^*(X(s), X(s+1)) = H^*(Y_s) = B_s$.

There is an isomorphism $\oplus_t E_1^{s,t} = \mathrm{Hom}_{\mathcal{A}}(B_s, \mathbb{Z}_p)$

Lemma: $d_1^{s,t}: E_1^{s,t} \to E_1^{s+1,t} = \mathrm{Hom}(B_s, \mathbb{Z}_p) \to \mathrm{Hom}(B_{s+1}, \mathbb{Z}_p)$, induce by $B_{s+1} \to B_s$

Pf: $B_{s+1} \to B_s$. induce by $Z_i = Y_{i+1}$, just consider the triple $(X(s), X(s+1), X(s+2))$

(of cohomology. check the connecting homomorphism)

Then. by the lemma + def of Ext, we just need proof b). to finish this.

Pf of b):

Idea: construct a homomorphism: $\mathrm{Ker}\, d_r^{s,t} \to E_{r+1}^{s,t}$, check $\begin{cases} \text{well def} \\ \text{onto} \\ \mathrm{Ker} = \mathrm{Im}(\cdots) \end{cases}$

$\alpha \in E_r^{s,t} \Rightarrow \alpha \in \pi_{N+t-s}(X(s+1-r), X(s+1))$

$F: (D^{N+t-s}, S^{N+t-s-1}) \to (X(s), X(s+r))$, It should be rep by $(\cdots) \to (X(s), X(s+r+1))$

Then, use $\alpha \in \operatorname{Ker} d_r^{s,t}$, $\Rightarrow \alpha$ annihilated by $\pi_{N+t-s}(X(s+1-r), X(s+1)) \to \pi_{N+t-s-1}(X(s+1), X(s+r+1))$

$\Rightarrow F|_{S^{N+t-s-1}}$ homotopic to a map into $X(s+r+1)$ $\Longrightarrow$ well def ✓

② onto (omit ... similiar to the above proof)

③ $\alpha \in \ker \Leftarrow$ The spheroid corr. to α homotopic to a trivial ~.

$H: D^{N+t-s} \times I \to X(s-r)$. F at the bottom, the top is in $X(s+1)$.

First, let $\alpha \in \operatorname{Ker} d_r^{s,t}$. As we have seen before, this means that α is an element of $\pi_{N+t-s}(X(s+1-r), X(s+1))$ representable by a spheroid $F\colon (D^{N+t-s}, S^{N+t-s-1}) \to (X(s), X(s+r))$. Furthermore, α belongs to the kernel of the map $\operatorname{Ker} d_r^{s,t} \to E_{r+1}^{s,t}$ if and only if F, regarded as a spheroid of the pair $(X(s-r), X(s+1))$, is homotopic to the trivial spheroid. This means that there exists a homotopy $H\colon D^{N+t-s} \times I \to X(s-r)$ which is F on the lower base and maps the side surface and the upper base into $X(s+1)$ (see Fig. 123). This map H may be regarded as a relative $(N+t-s+1)$-dimensional spheroid of the pair $(X(s-r), X(s))$. This spheroid determines an element of $\pi_{N+t-s-1}(X(s-r), X(s))$ whose image in $\pi_{N+t-s+1}(X(s+1-2r), X(s+1))$ is some element β of $E_r^{s-r,t-s+1}$. Modulo $X(s+1)$, the boundary of the spheroid H is F. Thus, $d_r^{s-r,t-s+1}\beta = \alpha$, and $\alpha \in \operatorname{Im} d_r^{s-r,t-s+1}$, as required.

ker ⊂ Im

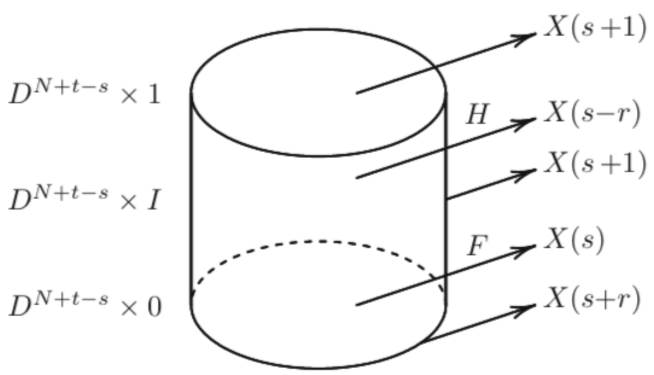


Fig. 123 A homotopy $H\colon (D^{N+t-s} \times I, S^{N+t-s-1} \times I) \to (X(s-r), X(s+1))$ of $F\colon (D^{N+t-s}, S^{N+t-s-1}) \to (X(s), X(s+r))$

+ Im ⊂ ker. (trivial)

Pf of 3) and 4) (The case of finite stable htpy gp.)

Suppose all of the stable htpy gp are finite in this section.

Lemma 1: $comp_{\bar{p}}\, \pi_{N+q}(X(s))$ does not depend on s.

Pf: Consider the htpy sequence induce by
$$\begin{array}{ccc} X(s+1) & \longrightarrow & X(s) \\ & & \downarrow \\ & & Y_s \end{array}$$

Lemma 2: For $q<n$, $\exists s_0$ s.t. $comp_p\, \pi_{N+q}(X(s)) = 0$. for $s > s_0$

Pf: Suppose m is the first number s.t. $comp\ H_{N+m}(X(s))$ nontrivial.

$X(s) \to Y_s$ induce an epimorphism of π_{N+m} (it induce $\oplus \mathbb{Z}_{p^{r_i}} \to \oplus \mathbb{Z}_p$.

And $comp_p\, \pi_{N+m}(X(s))$ is the kernel of it. $\square$.

Lemma 3. 1) $\pi_{N+q}(X(s), X(s'))$ is a p-group

2) s' large enough $\Rightarrow$ $\pi_{N+q}(X(s), X(s')) \cong comp_p\, \pi_{N+q}(X(s))$

Pf: $X(s') \hookrightarrow X(s)$ induce an exact sequence

non-p-component vanish by Lemma 1 $\quad$ 1) $\square$

$0 \to \pi_{N+q}(X(s)) \to \pi_{N+q}(X(s), X(s')) \to 0$ $\quad$ isomorphism by 2).

How to apply it?

$B^{s,t} \triangleq Im(\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(X(0)) = \pi^S_{t-s}(X)]$

check $E_\infty^{s,t} = B^{s,t}/B^{s-1,t-1}$ $\Longrightarrow$ For $E_M^{s,t}$, M large enough. + Lemma 3.
$E_M^{s,t}$ doesn't depend on M. So it is, actually, $E_\infty^{s,t}$.

$(E_M^{s,t} = Im[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))])$

This does not depend on M, so this is, actually, $E_\infty^{s,t}$. The homomorphism $\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))$ is a composition

$$\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))],$$

where the kernel of the second homomorphism is $\mathrm{Im}[\pi_{N+t-s}(X(s+1)) \to \pi_{N+t-s}(\Sigma^N X)]$, which is contained in the image of the first homomorphism. Hence,

$$\begin{aligned} E_\infty^{s,t} &= \frac{\mathrm{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X)]}{\mathrm{Ker}[\pi_{N+t-s}(\Sigma^N X) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]} \\ &= \frac{\mathrm{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X)]}{\mathrm{Im}[\pi_{N+t-s}(X(s+1)) \to \pi_{N+t-s}(\Sigma^N X)]} = \frac{B^{s,t}}{B^{s+1,t+1}}, \end{aligned}$$

which is statement (3) of the Adams theorem. Furthermore, for M large, by Lemmas 1 and 2,

$$\begin{aligned} B^{s+M,t+M} &= \mathrm{Im}[\pi_{N+t-s}(X(s+M)) \to \pi_{N+t-s}(\Sigma^N X)] \\ &= \mathrm{comp}_{\overline{p}}\, \pi_{N+t-s}(\Sigma^N X) = \mathrm{comp}_{\overline{p}}\, \pi_{t-s}^S(X), \end{aligned}$$

which is statement (4) of the Adams theorem.

This completes the proof of the Adams theorem in the case of finite stable groups. It remains to prove statements (3) and (4) in the general case. This requires some preparation.

Next question: what about the situation that we have $\mathbb{Z}$ in π_*^S?

To get this, we need $f: X \to X'$ induce a map between Adams filtrations and Adams SS.

①
$$\begin{tikzcd} \cdots \arrow[r] & B_0 \arrow[r] & \tilde{H}^*(X) \\ \cdots \arrow[r] & B_0' \arrow[r] \arrow[u, "\varphi_0"] & \tilde{H}^*(X') \arrow[u] \end{tikzcd}$$
$\Downarrow$

$X(0) \to X'(0)$ + $Y_0 \to Y_0'$ (induced by φ_0), since Y_0 is $\prod K(--)$

decides $X(1) \to X'(1)$, repeat this.

In the remain case, $Comp_p G$ contains the $\mathbb{Z}$ part.

Lemma 1 ✓

Lemma 2 need to be modify: If the order of $\alpha \in \pi_{N+q}(X(s))$ is ∞. or power of p,

M large enough, α does not belong to the image of the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$

Idea: $\Sigma X \xrightarrow{h = p^k\text{-fold}} \Sigma X$, Consider $X' = C(h)$

Consider the $\Sigma^N X \hookrightarrow \Sigma^{N-1} X'$ and the long htpy exact seq.

$(\pi_{N+q+1}(\Sigma^{N-1}X', \Sigma^N X) = \pi(--/--) = \pi_{N+q+1}(\Sigma^{N+1}X) = \pi_{N+q}(\Sigma^N X)$

But the connecting map ∂ is a p^k-fold map $\Rightarrow \pi_{N+q}(\Sigma^{N-1}X')$ p-group.

And the coker and ker of multiplication by p^k are finite p-group.

So $\pi_{N+q}(\Sigma^{N-1}X')$ finite p-group, $\pi_*^S(X')$ are finite.

And $\Sigma^N X \to \Sigma^{N-1} X'$ induce map in Adams filtration:

Suppose k large enough. s.t. $p^k \nmid ord(\alpha)$. But $ker(\pi_{N+q}(\Sigma^N X) \to \pi_{N+q}(\Sigma^{N-1}X'))$ consist of elements divided by p^k

$\Rightarrow \alpha \notin ker(\pi_{N+q}(\Sigma^N X) \to \pi_{N+q}(\Sigma^{N-1}X'))$, Its image not zero.

Apply the finite version to X', $\Rightarrow M$ large enough, $\beta \notin Im\ \pi_{N+q}(\Sigma^N X(M)) \to \pi_{N+q}(\Sigma^{N-1}X')$

Then $\alpha \notin Im(\pi_{N+q}(X(M)) \to \pi_{N+q}(\Sigma^N X)) \Rightarrow$ Lemma 2's modify have been proven.

Then, we just need to change proof by

$$\bigcap_{M \text{ large}} E_M^{s,t} = \bigcap_{M \text{ large}} Im(\pi_{N+t-s}(X(s), X(s+M)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1)) \cong Im[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))$$