Note of BKWX

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1 Introduction

(In the following discussion below we always omit $[1/e]$ from the notation. The serial numbers correspond to the serial numbers in BKWX unless I mention its source.)

Briefly speaking, [BKWX] is the generalization of [GWX] for the non-cellular case. (That is, the whole motivic stable homotopy category, instead of only the cellular part in GWX.) as well as for other base fields than just $\mathbb{C}$. To achieve this target, They built a t-structure called Chow t-structure on the motivic stable homotopy category $\mathcal{SH}(k)$ over any base field k.

The non-negative part of such t-structure which is called $\mathcal{SH}(k)_{c\geq 0}$ is generated under colimits and extensions by Thom spectra $Th(\xi)$ associated to K-theory points $\xi \in K(X)$ (equivalently formal differences of vector bundles $\xi = [V_1] - [V_2]$) on smooth and proper schemes X (see Definition 2.1). We denote the truncation functors by $E \mapsto E_{c=0}, E_{c\geq 0}$, and so on. Here, we use Thom spectra since they are models of MGL. Refer to Appendix A. Reasons are shown in the following paragraph.

Like what the authors did in [GWX], the topological part and the algebraic part are connected by a spectral sequence in their hearts. The motivic bigraded homotopy groups of a homotopy object with respect to the Chow t-structure can be expressed by Ext groups over $MU_{2*}MU$ of its MGL homology (Theorem 1.1). This theorem allows us to build an equivalence between the Chow heart $\mathcal{SH}(k)^{\heartsuit}$ and something on the algebraic side (Theorem 1.5). Restricting to some subcategory of (W-)cellular of modules of some E_∞-rings in $\mathcal{SH}(k)$, we get some equivalences like to the results in GWX.

For the proof of Theorem 1.5, BKWX follows the following strategy. For a motivic commutative ring spectrum $A \in CAlg(\mathcal{SH}(k))$, there is an induced Chow t-structure on the category $A\text{-}Mod$ (just let the non-negative part be generated by free A-modules of the form $A \wedge Th(\xi)$, for X smooth proper and $\xi \in K(X)$). We focus on the induced t-structures on $\mathbb{1}_{c=0}\text{-}Mod$ and also $MGL_{c=0}\text{-}Mod$. The free-forgetful adjunction

$$\mathbb{1}_{c=0}\text{-}Mod \leftrightarrows MGL_{c=0}\text{-}Mod \tag{1}$$

defines a comonad C over $MGL_{c=0}\text{-}Mod$. We show that when restricted to subcategories of suitably bounded objects, the adjunction is comonadic. Then with the technical developed in section 4.1,

$\mathbb{1}_{c=0}$ - $Mod^{c\heartsuit}$ is equivalent to the category of $C^{c\heartsuit}$-comodules in $MGL_{c=0}$ - $Mod^{c\heartsuit}$, where C - $CoMod$ has a natural t-structure which have some relations with the induced Chow t-structure of A - Mod and $C^{c\heartsuit}$ denotes the restriction of C in its heart.

Using the spectral Morita theory discussed in section 4.2, we can identify the category $MGL_{c=0}$ - Mod as a presheaf category which values in $MU_{2*}MU$ - $CoMod$. Both of them have t-structure and the equivalence is t-exact. The information of an object in the heart of $MGL_{c=0}$ - Mod is equivalent to some object in $C^{c\heartsuit}$ - $CoMod$. With the result $\mathcal{SH}(k)^{c\heartsuit} \simeq \mathbb{1}_{c=0}$ - $Mod^{c\heartsuit}$, we get the Theorem 1.5.

Remark. According to the work of Bondarko [Bon10], $MGL_{c=0}$ - $Mod^{c\heartsuit} \simeq MGL$ - $Mod^{c\heartsuit}$ has an explicit description. This proof is equivalent to what we summarized above as "spectral Morita theory".

For the proof of Theorem 1.1, BKWX follows the following strategy. The main idea is to use a series of t-structure as filtration. To achieve this target, the authors of BKWX use another t-structure on $\mathcal{SH}(k)$, the homotopy t-structure. Its non-negative part $\mathcal{SH}(k)_{\geq 0}$ is generated under colimits and extensions by $\{\Sigma^{\infty}_{+} X \wedge \mathbb{G}_m^{\wedge n} | n \in \mathbb{Z}, x \in Sm_k\}$. For all $d \geq 0$, the intersections

$$I^d := \Sigma^d \mathcal{SH}(k)_{\geq 0} \cap \mathcal{SH}(k)_{c\geq 0} \tag{2}$$

define a sequence of further t-structures. Here I is a subset in $\mathcal{SH}(k)_{c\geq 0}$ with some restriction which decides a t-structure. We write $\tau^d_{=0}$ for the 0-th truncation functor with respect to I^d. It turns out that these t-structures form a direct system with the Chow t-structure as the colimit. Of course, the "colimit" of t-structures will be discussed later.

One key property of these t-structures is the following vanishing result.

Proposition 1.1. *Let $E \in \mathcal{SH}(k)[1/e]$.*

(1) $\pi_{*,*}\tau^d_{=0}E$ is concentrated in Chow degrees ≤ 0, and

(2) $MGL_{2*,*}\tau^d_{=0}E$ equals $MGL_{2*,*}E$ and vanishes for other bidegrees.

As a result, the Adams Novikov spectral sequence for $\tau^d_{=0}E$ collapses and converges to $\pi_{*,*}\tau^d_{=0}E$. The canonical isomorphism can be written as

$$\pi_{2w-s,w}(\tau^d_{=0}E) \cong Ext^{s,2w}_{MU_*MU}(MU_*, MGL_{2*,*}E) \tag{3}$$

Since we can approximate $E_{c=0}$ as a colimit by $\tau^d_{=0}E$, the same result holds for the Chow t-structure. This is exactly the statement of Theorem 1.1.

In BKWX, Theorem 1.3 in [GWX] (the equivalence between the motivic Adams spectral sequence for $S^{0,0}/\tau$ and the algebraic Novikov spectral sequence for BP_{2*}.) is generalized to Theorem 1.18 in BKWX. Just like the result in GWX, Theorem 1.18 can help us to decide some differentials in the motivic Adams spectral sequence.

I will list those theorems I just mentioned in the above paragraphs as follows.

Theorem 1.2 (1.1)**.** *Let $E \in \mathcal{SH}(k)[1/e]$. Then there is a canonical isomorphism*

$$\pi_{2w-s,w}E_{c=i} \cong Ext^{s,2w}_{MU_{2*}MU}(MU_{2*}, MGL_{2*+i,*}E). \tag{4}$$

Here, on the right-hand side, s is the homological degree, and $2w$ is the internal degree.

Remark: $\mathbb{1}_{c=0}$ and $\mathbb{1}_p^\wedge/\tau$ are E_∞-rings over $\mathbb{C}$.

Definition 1.1 (1.4)**.** The category of pure MGL-motives, denoted by $PM_{MGL}(k)$, is the smallest idempotent complete additive subcategory of $MGL_{c=0}\text{-}Mod$ containing the object $X\{i\} := (\Sigma_{2*,*}X_+ \wedge MGL)_{c=0}$ for each $i \in \mathbb{Z}$ and smooth proper variety X.

Theorem 1.3 (1.5)**.** *The functor sending $F \in \mathcal{SH}(k)^{c\heartsuit}[1/e]$ to the presheaf on $PM_{MGL}(k)$ given by $F_*(X) = [\Sigma 2*, *X_+, F \wedge MGL]$ induces an equivalence of categories between $\mathcal{SH}(k)^{c\heartsuit}[1/e]$ and the category of enriched presheaves on $PM_{MGL}(k)$ (with values in $MU_{2*}MU$-comodules).*

Theorem 1.4 (1.8)**.** *There is a symmetric monoidal equivalence*

$$\mathbb{1}_{c=0}\text{-}Mod[1/e] \simeq Hov(\mathbb{1}_{c=0}\text{-}Mod[1/e]^{c\heartsuit}). \tag{5}$$

Theorem 1.5 (1.9)**.** *(1) There is an equivalence*

$$\mathcal{SH}(k)[1/e]^{cell,c\heartsuit} \simeq MU_{2*}MU = CoMod[1/e]. \tag{6}$$

(2) The cellular subcategory is equivalent to Hovey's stable category of comodules

$$\mathbb{1}_{c=0}\text{-}Mod^{cell}[1/e] \simeq Hov(MU_{2*}MU)[1/e]. \tag{7}$$

Theorem 1.6 (1.18)**.** *Let $F \in \mathcal{SH}(k)^{c\heartsuit}$. Let $M = MGL_{2*,*}F$ be the associated $MU_{2*}MU$-comodule. Then the trigraded motivic Adams spectral sequence for F based on $H\mathbb{Z}/p$ is isomorphic (with all higher and multiplicative structures) to the trigraded algebraic Novikov spectral sequence based on H for M.*

Here the H is got by the correspondence in Theorem 1.9 of the element $H\mathbb{Z}/p \wedge \mathbb{1}_{c=0}$, that is, $MU_{2*}MU/(p, a_1, a_2, \cdots)$.

2 Elementry properties of the Chow t-structure.

In GWX, the authors found that the category $\mathbb{1}_p^\wedge/\tau\text{-}Mod$ of cellular modules over $\mathbb{1}_p^\wedge/\tau$ is purely algebraic. That is, the heart is equivalent to $MU_{2*}MU_p^\wedge\text{-}CoMod$ and the information of the whole module category can be recovered by Hovey's stable category of $MU_{2*}MU_p^\wedge$. In BKWX, this result is generalized to an integral result over arbitrary fields, using $\mathbb{1}_{c=0}$ as a replacement for $\mathbb{1}_p^\wedge/\tau$. The $\mathbb{1}_{c=0}$ is decided by the Chow t-structure of $\mathcal{SH}(k)$, which have some good properties that we will introduce in this section.

In the lecture, we will just list the main results in this part without any explanation or remark in this part. The audience can refer to the original paper if they are interested in this part.

Suppose that S is a scheme.

Definition 2.1. Denote by $\mathcal{SH}(S)_{c\geq 0}$ the subcategory generated under colimits and extensions by motivic Thom spectra $Th(\xi)$ for X smooth and proper over S and $\xi \in K(X)$ arbitrary. This is the non-negative part of a t-structure on $\mathcal{SH}(S)$ [HA, Proposition 1.4.4.11(2)] called the Chow t-structure. We denote the non-positive part by $\mathcal{SH}(S)_{c\leq 0}$, the heart by $\mathcal{SH}(S)^{c\heartsuit}$ and write $E \mapsto E_{c\geq 0}, E_{c\leq 0}, E_{c=0}$ for the truncation functors. We also put $\mathcal{SH}(S)_{c\geq n} = \Sigma^n \mathcal{SH}(S)_{c\geq 0}$ and define $\mathcal{SH}(S)_{c\leq n}$, $E_{c\geq n}$ etc. similarly.

By some basic properties of t-structure, $E \in \mathcal{SH}(S)_{c\leq 0}$ if and only if $[\Sigma^i Th(\xi), E] = 0$ for all $i > 0$. We can also find the strong duals of $Th_S(\xi) \in \mathcal{SH}(S)$ (Lemma 2.5).

Here are some other properties: Chow t-structure is compatible with filtered colimits and the symmetric monoidal structure of $\mathcal{SH}(S)$ (Proposition 2.7, 2.9).

Proposition 2.1. *The Chow t-structure is compatible with filtered colimits: $\mathcal{SH}(S)_{c\leq 0}$ is closed under filtered colimits (and so are $\mathcal{SH}(S)_{c\geq 0}$ and $\mathcal{SH}(S)_{c=0}$).*

Corollary. *The functor $E \mapsto E_{c\leq 0} : \mathcal{SH}(S) \to \mathcal{SH}(S)$ preserves filtered colimits.*

Proposition 2.2. *We have*

$$\mathcal{SH}(S)_{c\leq 0} \wedge \mathcal{SH}(S)_{c\leq 0} \subset \mathcal{SH}(S)_{c\leq 0} \tag{8}$$

Definition 2.2. $\mathcal{SH}(S)^{pure} \subset \mathcal{SH}(S)_{c\leq 0}$ the smallest subcategory that is closed under filtered colimits and extensions and contains $Th(\xi)$ for any K-theory point ξ on a smooth proper scheme X over S.

Proposition 2.3. *We have*

$$\mathcal{SH}(S)^{pure} \wedge \mathcal{SH}(S)_{c\leq 0} \subset \mathcal{SH}(S)_{c\leq 0} \tag{9}$$

Corollary. *For $X \in \mathcal{SH}(S)^{pure}$, $Y \in \mathcal{SH}(S)$ we have $Y_{c\leq 0} \wedge X \simeq (Y \wedge X)_{c\leq 0}$, $Y_{c\geq 0} \wedge X \simeq (Y \wedge X)_{c\geq 0}$ and $Y_{c=0} \wedge X \simeq (Y \wedge X)_{c=0}$.*

Definition 2.3. $\mathcal{SH}(S)^{lisse}$ is the stable presentable subcategory generated by $\mathcal{SH}(S)^{pure}$.

Proposition 2.4. *The Chow t-structure on $\mathcal{SH}(S)^{lisse}$ is right complete (but not typically left complete).*

Definition 2.4. A spectrum $E \in \mathcal{SH}(S)^{lisse}$ is Chow-∞-connective if $E \in \mathcal{SH}(S)_{c\geq n}$ for all n.

Proposition 2.5. *If $E \in \mathcal{SH}(S)^{lisse}$ is η-periodic, then E is Chow-∞-connective.*

Proposition 2.6. *The Chow t-structure has some compatibility with the base change.*

Proposition 2.7. *The Chow t-structure has some compatibility with cellularization.*

3 Relationship to algebraic cobordism

The authors show that the Chow t-structure is the "colimit" of directed systems of t-structures (actually the directed systems of truncation functors) as we mentioned in the sketch of proof of Theorem 1.1. Since the MGL-homology groups of those objects in the heart of such t-structures have some good properties which are preserved by the colimit, the MGL-homology of the elements in the Chow t-structure can be computed by Adams-Novikov spectral sequence (actually it collapses at the E_2-page).

Here's some notation in this section. We let $S = Spec(k)$ be the spectrum of a field of exponential characteristic e, and we implicitly invert e throughout. Thus we write $\mathcal{SH}(k)$ for $\mathcal{SH}(k)[1/e]$, and so on. The main reason for this is that we need to use a vanishing result for algebraic cobordism (Theorem B.1) which is proved by relating algebraic cobordism to higher Chow groups, and this relationship is currently only known away from the characteristic [Hoy15].

We can choose an $I \subset \mathcal{SH}(k)_{c\geq 0}$ which is s closed under colimits and extensions and generated by a set of compact objects. We can generate a t-structure with I, the non-negative part is denoted as $\mathcal{SH}(k)_{I\geq 0}$, and other notations are decided in the same way.

Proposition 3.1. *Let $E \in \mathcal{SH}(k)_{I\geq 0}$ (resp. $E \in \mathcal{SH}(k)_{I\leq 0}$, $E \in \mathcal{SH}(k)_{I=0}$). Then*

(1) $MGL \wedge E \in \mathcal{SH}(k)_{I\geq 0}$ (resp. ...), and

(2) $MGL_{*,*}E$ is concentrated in Chow degrees ≥ 0 (resp. ...)

As a result, we know that the MGL-homology of $\tau_{I=i}E$ concentrated in Chow degree i, and

$$MGL_{p,q}\tau_{I=i}E = MGL_{p,q}(E), \quad if \quad c(p,q) = i. \tag{10}$$

Its proof can be done by applying MGL on the fiber sequence decided by τ.

After the setting of I, we can apply it to the homotopy t-structure.

Definition 3.1. Let $\mathcal{SH}(k)_{\geq 0}$ denote the subcategory of $\mathcal{SH}(k)$ generated under colimits and extensions by $\Sigma^\infty_+ X \wedge \mathbb{G}_m^{\wedge n}$ for $X \in Sm_k$ and $n \in \mathbb{Z}$. This defines the non-negative part of the homotopy t-structure.

According to [Hoy15, Theorem 2.3], we have

$$\mathcal{SH}(k)_{\geq 0} = \{E \in \mathcal{SH}(k) | \underline{\pi}_{i+n,n}(E) = 0 for i < 0, n \in \mathbb{Z}\}(resp. \leq 0) \tag{11}$$

Definition 3.2 (actually a theorem)**.** Let I^d be the category $\mathcal{SH}(k)_{\geq d} \cap I$. This is the non-negative part of a t-structure on $\mathcal{SH}(k)$.

Next, we need to show the fibers we need staying in the I^d and that their MGL-homologies of them aren't affected by this operation.

Proposition 3.2. *Let $E \in I^d$. Then we have the following properties:*

(1) $\tau^d_{\leq 0}E \in I^d$,

(2) $MGL_{2*,*}\tau^d_{\leq 0}E \cong MGL_{2*,*}E$,

Suppose now that $E \in I^{d+1}$. Then we have additionally the following:

(3) $\pi_{*,*}\tau^d_{\leq 0}E$ is concentrated in Chow degrees ≤ 0, and

(4) $MGL_{*,*}\tau^d_{\leq 0}E$ is concentrated in Chow degree 0.

In addition, the directed system along the degree of truncation functors forms a colimit diagram.

Proposition 3.3. *Let $E \in \mathcal{SH}(k)$. There are directed systems*

$$\tau^n_{\le 0}E \to \tau^{n-1}_{\le 0}E \to \cdots \to \tau_{\le 0}E, \tag{12}$$

$$\tau^n_{\ge 0}E \to \tau^{n-1}_{\ge 0}E \to \cdots \to \tau_{\ge 0}E, \tag{13}$$

which are colimit diagrams.

We have an MGL-based motivic Adams-Novikov spectral sequence as follows:

$$Ext^{*,*,*}_{MGL_{*,*}MGL}(MGL_{*,*}, MGL_{*,*}X) \Rightarrow \pi_{*,*}X^\wedge_{MGL} \tag{14}$$

$(MGL_{2*,*}, MGL_{2*,*}MGL)$ is a Hopf algebroid canonically isomorphic to $(MU_{2*}, MU_{2*}MU)$.

In particular for $E \in \mathcal{SH}(k)$, the graded abelian group $MGL_{2*,*}E$ is canonically a comodule over $MU_{2*}MU$.

Apply the MGL-based motivic Adams-Novikov spectral sequence on the above spaces, we have:

Theorem 3.4. *Let $E \in \mathcal{SH}(k)_{I\ge 0}$. Then*

(1) the canonical map $\tau_{I\le 0}E \to \tau_{I\le 0}E^\wedge_{MGL}$ to the MGL-nilpotent completion induces an isomorphism on $\pi_{*,*}$, and

(2) $\pi_{2w-s,w}\tau_{I\le 0}E \cong Ext^{s,2w}_{MU_*MU}(MU_*, MGL_{2*,*}E)$

The main idea of the proof is the following isomorphisms:

$$\tau_{I\le 0}E \simeq \tau_{I\le 0}\tau_{I\ge 0}E \simeq \tau_{I\le 0}colim\tau^d_{I\ge 0}E \simeq colim\tau_{I\le 0}\tau^d_{I\ge 0}E \simeq colim\tau^n_{I\le 0}\tau^d_{I\ge 0}E \tag{15}$$

Let $n < d$. We shall show that $\pi_{2w-s,w}\tau^n_{\le 0}E \simeq Ext^{s,2w}_{MU_*MU}(MGL_{2*,*}E)$ and we can get this result by considering the colimit of $n \to \infty$.

By Lemma 3.12 (2, 4), we find that $MGL_{2*+i,*}\tau^n_{\le 0}E = MGL_{2*,*}E$ for $i = 0$ and vanishes else. This leads to the convergence of the Adams-Novikov spectral sequence. Then, $\tau^n_{\le 0}E$ is connective in the homotopy t-structure; hence by [Man21, §5.1 and Theorem 7.3.5] we have

$$\tau^n_{\le 0}(E)^\wedge_{MGL} \simeq \tau^n_{\le 0}(E)^\wedge_\eta \tag{16}$$

Since thr connective property of $\tau^n_{\le 0}E$, $\tau^n_{\le 0}(E)^\wedge \to \tau^n_{\le 0}(E)^\wedge_\eta$ induces an equivalence on $\pi_{*,*}$. This concludes the proof of part (2).

For part (1), Corollary 3.7 implies that the Adams-Novikov spectral sequence for $\tau^{I\le 0}E$ collapses and

$$\pi_{2w-s,w}\tau_{I\le 0}(E)^\wedge_{MGL} \cong Ext^{s,2w}_{MU_*MU}(MU_*, MGL_{2*,*}E) \tag{17}$$

Hence part (1) follows from part (2).

Here are some corollaries of this theorem that we will use later.

Lemma 3.5. *Let $E \in \mathcal{SH}(k)$. If $\pi_{i,0}(E \wedge Th(\xi)) = 0$ for all $i \in \mathbb{Z}$ and K-theory points ξ on smooth proper varieties X, thn $E \simeq 0$.*

Corollary. *Let $E \in \mathcal{SH}(k)$. Then E is Chow-∞-connective if and only if $E \wedge MGL \simeq 0$.*

4 Reconstruction theorem

4.1 Comonadic descent

By [Bac18b, Lemma 29], we know that $\mathbb{1}_{c=0}\text{-}Mod^{c\heartsuit} \simeq \mathcal{SH}(k)^{c\heartsuit}$. As a result, we can get the information of $\mathcal{SH}(k)^{c\heartsuit}$ by analysising the structure of $\mathbb{1}_{c=0}\text{-}Mod^{c\heartsuit}$. The tool we will use is the comonadic descent.

Suppose given a presentably symmetric monoidal category $\mathcal{D}$ and $A \in CAlg(\mathcal{D})$. We obtain a free-forgetful adjunction

$$F : \mathcal{D} \leftrightarrow A\text{-}Mod : U \tag{18}$$

and the endofunctor

$$C := FU = \otimes A : A\text{-}Mod \to A\text{-}Mod \tag{19}$$

which acquires the structure of a comonad.

We denote by $C\text{-}CoMod$ the category of comodules under C [Lur17a, Definition 4.2.1.13], and hence obtain a factorization [Lur17a, §4.7.4]

$$\mathcal{D} \leftrightarrow C\text{-}CoMod \leftrightarrow A\text{-}Mod \tag{20}$$

where $C\text{-}CoMod \to A\text{-}Mod$ is the forgetful functor which we denote by H (with right adjoint the cofree comodule functor) and $\mathcal{D} \to C\text{-}CoMod$ sends X to $X \otimes A$ with its canonical comodule structure.

We can use cobar resolution to describe those objects in $C\text{-}CoMod$ in the classical way. Its monoidal structure is induced by the left adjunction and its t-structure is induced by the cobar resolution. That is, the subcategory corresponds to $limCB^{\bullet}(A)\text{-}Mod_{[m,n]}$ which means the subcategory of those chain complexes bounded in this range. We denote such category as $C\text{-}CoMod_{[m,n]}$. For $C_{[m,n]} : A\text{-}Mod_{[m,n]} \to A\text{-}Mod_{[m,n]}$, we have $C\text{-}CoMod_{[m,n]} \simeq C_{[m,n]}\text{-}CoMod$. $C^{\heartsuit} = C_{[0,0]}$. $[0, \infty]$ and $[-\infty, 0]$ correspondent to the non-negative part and the non-positive part.

Then, we can try to describe the heart of such ∞-category with t-structure. Let $C\text{-}CoMod^{\heartsuit\omega}$ denote the category of its compact objects. According to Hovey's result, we have

$$Hov(C) := Ind(Thick(C\text{-}CoMod^{\heartsuit\omega})) \tag{21}$$

where $Thick(C\text{-}CoMod^{\heartsuit\omega})$ denotes the thick subcategory of $C\text{-}CoMod$ generated by $C\text{-}CoMod^{\heartsuit\omega}$ and Ind denote the category obtained by freely adding filtered colimits (see [Lur17b, 5.3.5.1]), which have an adjunction

$$Hov(C) \leftrightarrow C\text{-}CoMod. \tag{22}$$

By setting $\mathcal{D} = \mathbb{1}_{c=0}\text{-}Mod$ and $A = MGL \wedge \mathbb{1}_{c=0}$ (just $MGL_{c=0}$), we have such adjunctions:

$$\mathbb{1}_{c=0}\text{-}Mod \leftrightarrow C\text{-}CoMod \leftrightarrow MGL_{c=0}\text{-}Mod. \tag{23}$$

For the Chow t-structure in $\mathbb{1}_{c=0}\text{-}Mod$ and the t-structure we defined as above, we have

Proposition 4.1. *(1) The free functor* $\overline{F} : \mathbb{1}_{c=0}\text{-}Mod \to C\text{-}CoMod$ *is* t*-exact and symmetric monoidal.*

(2) For $-\infty \leq m \leq n < \infty$*, the restriction* $\mathbb{1}_{c=0}\text{-}Mod_{[m,n]} \to C\text{-}CoMod_{[m,n]}$ *is an equivalence. In particular*

$$\mathbb{1}_{c=0}\text{-}Mod^{c\heartsuit} \simeq C\text{-}CoMod^{c\heartsuit}. \tag{24}$$

(3) The functor $\overline{F}$ *induces a symmetric monoidal equivalence* $\mathbb{1}_{c=0} \simeq Hov(C)$*.*

For the second result, we need the Barr-Beck-Lurie theorem [Lur17a, Theorem 4.7.3.5]. This theorem need us to prove: $\mathbb{1}_{c=0}\text{-}Mod_{[m,n]} \to MGL_{c=0}\text{-}Mod_{[m,n]}$. is conservative, and F-split totalizations exist in $\mathbb{1}_{c=0}\text{-}Mod_{[m,n]}$ and are preserved by F.

Then, we have

$$\mathcal{SH}(k)^{c\heartsuit} \simeq \mathbb{1}_{c=0}\text{-}Mod^{c\heartsuit} \simeq C^{c\heartsuit}\text{-}CoMod. \tag{25}$$

where $C^{c\heartsuit}$ is the comonad on $MGL_{c=0}\text{-}Mod^{c\heartsuit}$ obtained by restricting C.

4.2 Spectral Morita theory

We still need to deal with the right part of the following adjunctions.

$$\mathcal{D} \leftrightarrow C\text{-}CoMod \leftrightarrow A\text{-}Mod \tag{26}$$

that is, connect $A\text{-}Mod$ with $C\text{-}CoMod$ in a heart-preserving way. In addition, we need to connect $A\text{-}Mod$ with something on the algebraic side. All of these are finished by the spectral Morita theory.

Roughly speaking, the spectral Morita theory is a category of presheaf over the pure MGL-motives which have some extra structure. The algebraic information is hidden in the extra structure. (The extra structure is the mapping space between each object in $PM_{MGL}(k)$ is enriched over $MU_{2*}MU\text{-}CoMod$.)

For a smooth proper variety X and $i \in \mathbb{Z}$, denote $X\{i\} \in MGL_{c=0}\text{-}Mod$ as the object $(\Sigma^{2i,i}X_+ \wedge MGL)_{c=0} \simeq \Sigma^{2i,i}(X_+)_{c=0} \wedge MGL$. By the Thom isomorphism, these are generators of $MGL_{c=0}\text{-}Mod$ (as a localizing subcategory). Write $PM_{MGL}(k) \subset MGL_{c=0}\text{-}Mod$ for the smallest idempotent complete additive subcategory containing the objects $X\{i\}$. For now, we view this as a spectrally enriched category. By duality and adjunction, we have

$$\begin{aligned} &Map_{PM_{MGL}(k)}(X\{i\}, Y\{j\}) \\ \simeq &Map_{PM_{MGL}(k)}(MGL_{c=0}, (X \times Y)\{j - d_X - i\}) \\ \simeq &Map_{\mathcal{SH}(k)}(S^{2(i+d_X-j),(i+d_X-j)}, ((X \times Y)_+)_{c=0} \wedge MGL). \end{aligned} \tag{27}$$

$$\pi_* Map_{PM_{MGL}(k)}(X\{i\}, Y\{j\}) \simeq MGL_{*+2(i+d_X-j),(i+d_X-j)}((X \times Y)_+)_{c=0} \tag{28}$$

is concentrated in degree 0 due to proposition 3.6 (That proposition describes the MGL-homology.) In other words, our spectrally enriched category PM_{MGL}(k) is just an additive ordinary 1-category. The above computation together with Corollary 3.7 shows that

$$Map_{PM_{MGL}(k)}(X\{i\}, Y\{j\}) \simeq MGL_{2(i+d_X-j),(i+d_X-j)}(X \times Y) \simeq MGL^{2(d_Y+j-i),(d_Y+j-i)}(X \times Y). \tag{29}$$

Remark. We can generalize this result by replacing MGL with another oriented ring spectrum in $\mathcal{SH}(k)$. It also has some compatibility with maps between such spectra.

Since we have an explicit description of $PM_{MGL}(k)$ as a spectrally enriched category, we should be able to recover $MGL_{c=0}\text{-}Mod$ by a variant of Morita theory, such as [SS03].

For an ∞-category $\mathcal{D}$ with finite coproducts, we use the notation

$$\mathcal{P}_\Sigma(\mathcal{D}) = Fun^\times(\mathcal{D}^{op}, Spc), \mathcal{P}_{\mathcal{SH}}(\mathcal{D}) = Fun^\times(\mathcal{D}^{op}, \mathcal{SH}), \mathcal{P}_{Ab}(\mathcal{D}) = Fun^\times(\mathcal{D}^{op}, Ab), \tag{30}$$

where $Fun^\times$ denotes the category of product-preserving functors. Provided that $\mathcal{D}$ is additive, there are equivalences [Lur18, Remark C.1.5.9]

$$\mathcal{P}_{\mathcal{SH}}(\mathcal{D})_{\geq 0} \simeq \mathcal{P}_\Sigma(\mathcal{D}), \mathcal{P}_{\mathcal{SH}}(\mathcal{D})^{\heartsuit} \simeq \mathcal{P}_{Ab}(\mathcal{D}). \tag{31}$$

Lemma 4.2. *Let $\mathcal{D}$ be a small semi-additive ∞-category. The full subcategory $\mathcal{P}_{\mathcal{SH}}(\mathcal{D})_{\geq 0}$ consisting of functors $F : \mathcal{D}^{op} \to \mathcal{SH}_{\geq 0} \subset \mathcal{SH}$ is generated under colimits and extension by the image of the canonical functor $\mathcal{D} \to \mathcal{P}_{\mathcal{SH}}(\mathcal{D})$. In particular $\mathcal{P}_{\mathcal{SH}}(\mathcal{D})_{\geq 0}$ is the non-negative part of a t-structure. Its non-positive part consists of the functors $F : \mathcal{D}^{op} \to \mathcal{SH}_{\leq 0} \subset \mathcal{SH}$.*

We have such equivalence between ∞-categories with t-structure:

Proposition 4.3. *$\mathcal{P}_{\mathcal{SH}}(PM_{MGL}(k)) \simeq MGL_{c=0}\text{-}Mod$ is a canonical t-exact, symmetric monoidal equivalence.*

As a result, we can define $\underline{\pi}_i^c(E)(X) = \pi_i Map_{MGL_{c=0}\text{-}Mod}(X, E)$ for given $E \in MGL_{c=0}\text{-}Mod$ which induces an equivalence $MGL_{c=0}\text{-}Mod^{c\heartsuit} \to \mathcal{P}_{Ab}(PM_{MGL}(k))$ when $i = 0$.

Then, the equivalence can be described by such explicit functors:

Corollary. *(1) The functor $\underline{\pi}_0^c : MGL_{c=0}\text{-}Mod^{c\heartsuit} \to \mathcal{P}_{Ab}(PM_{MGL}(k))$ is an equivalence.*

(2) For $E \in MGL_{c=0}\text{-}Mod$ we have $E \in MGL_{c=0}\text{-}Mod_{c\geq 0}$ iff $\underline{\pi}_i^c(E) = 0$ for all $i < 0$. (It's the same for other parts in a t-structure)

(3) The Chow t-structure on $MGL_{c=0}\text{-}Mod$ is non-degenerate.

An object $F \in MGL_{c=0} - Mod^{c\heartsuit} \simeq \mathcal{P}_{Ab}(PM_{MGL}(k))$ has such information according to the above equivalence:

- For every smooth proper variety X a graded abelian group $F(X)_* = \underline{\pi}_0^c(F(X\{*\}))$
- For every graded MGL-correspondence $\alpha : X \to Y$ a homomorphism $\alpha^* : F(Y)_* \to F(X)_*$, subject to the conditions that
  - for composable MGL-correspondences α, β we have $\alpha^*\beta^* = (\beta\alpha)^*$,

– $id^* = id$ and $0^* = 0$, as well as

– for parallel MGL-correspondences $\alpha^* + \beta^* = (\alpha+\beta)^*$

Since$\mathcal{P}_{Ab}(PM_{MGL}(k))$ has enough projective objects, with the help of Lurie's theorem, we have $\mathcal{P}_{\mathcal{SH}}(PM_{MGL}(k)) \simeq D(\mathcal{P}_{Ab}(PM_{MGL}(k)))$.

The diagram in the P20 of BKWX, draw it in the formal version.

Finally, we need to connect the above discussion with the comonad we discussed before.

Definition 4.1. $Fun^L_0(MGL_{c=0}\text{-}Mod, MGL_{c=0}\text{-}Mod)$ is a subcategory of those functors F which preserve colimits and heart of $MGL_{c=0}\text{-}Mod$.

Lemma 4.4. *$Fun^L_0(MGL_{c=0}\text{-}Mod, MGL_{c=0}\text{-}Mod) \to Fun^L(MGL_{c=0}\text{-}Mod^{c\heartsuit}, MGL_{c=0}\text{-}Mod^{c\heartsuit})$ is an equivalence.*

This lemma implies that {cocontinuous comonads on $MGL_{c=0}\text{-}Mod$ preserving the heart} equals to {cocontinous comonads on $MGL_{c=0}\text{-}Mod^{c\heartsuit}$}. Under the above equivalence, the comonad C corresponds to its restriction to the heart, which we denote by $C^{c\heartsuit}$ or also by C when it is clear in the context. Now, we need to check that the elements in $C^{c\heartsuit}\text{-}CoMod$ correspondent to some element in $\mathcal{P}_{Ab}(PM_{MGL}(k))$. That is, check C preserves the heart.

We can view $[X\{*\}, Y\{*\}]$ as a single graded group by taking the first $*$ to be 0. Observe that for smooth proper varieties X, Y, the mapping set, $[X\{0\}, Y\{*\}]_{PM_{MGL}(k)}$, is an $MU_{2*}MU$-comodule; indeed we have seen that up to some shift in degrees it identifies with $MGL^{2*,*}(X\times Y)$. In other words, for any graded MGL-correspondence $\alpha : X \to Y \in [X\{0\}, Y\{*\}]_{PM_{MGL}(k)}$, we obtain

$$\triangle(\alpha) = \sum_i p_i \otimes \alpha_i \in MU_{2*}MU \otimes_{MU_{2*}} MGL_{2*,*}(X\times Y) \simeq MU_{2*}MU \otimes_{MU_{2*}} [X\{0\}, Y\{*\}]_{PM_{MGL}(k)} \tag{32}$$

For $F \in MGL_{c=0}\text{-}Mod^{c\heartsuit}$, due to Remark 4.10, is a kind of presheaf on smooth proper varieties together with some extra data, namely an action by MGL-correspondences. We wish to describe $CF \in MGL_{c=0}\text{-}Mod^{c\heartsuit}$, again this is a presheaf with an action by MGL-correspondences.

Remark. We have $CF = MGL_{c=0} \wedge_{1c=0} F$, which has two structures as an $MGL_{c=0}$-module. Since the underlying spectra are the same, the right module structure has the same value on sections as the left module structure when viewed as an object of $\mathcal{P}_{\mathcal{SH}}(PM_{MGL}(k))$, however, the action by graded MGL-correspondences differs. The "correct" action is on the left, and given a correspondence $\alpha : X \to Y$ we denote it by $\alpha^*_L : CF(Y) \to CF(X)$

Proposition 4.5. *(1) Given $F \in MGL_{c=0}\text{-}Mod^{c\heartsuit} \simeq \mathcal{P}_{Ab}(PM_{MGL}(k))$, the object CF is given on sections by $(CF)(X)_* = MU_{2*}MU \otimes_{MU_{2*}} F(X)_*$. Given an MGL-correspondence $\alpha : X \to Y$, the action $\alpha^*_L : CF(Y) \to CF(X)$ is given by $\triangle(\alpha)^*$. In other words for $s \in F(Y)$and $p \in MU_{2*}MU$ we have $\alpha^*_L(p\otimes s) = \sum_i pp_i \otimes \alpha^*_i(s)$, in the notation for $\triangle(\alpha)$ of above.*

(2) The counit map $CF \to F$ is given on sections by $p \otimes s \to \epsilon(p)s$, where ϵ is the counit of the Hopf algebroid $(MU_{2*}, MU_{2*}MU)$.

(3) The comultiplication map $CF \to C^2F$ is given on sections by $p \otimes s \to \triangle(p) \otimes s$.

Remark. As we have observed above, the mapping sets in $PM_{MGL}(k)$ are naturally $MU_{2*}MU$-comodules. In fact this makes $PM_{MGL}(k)$ into a category enriched in $MU_{2*}MU$-comodules. The compatibility condition displayed above precisely means that $\mathcal{SH}(k)^{c\heartsuit}$ is equivalent to the category of enriched presheaves on $PM_{MGL}(k)$ (see e.g. [Rie14, §3.5]).

Here is the second main result of this paper:

Theorem 4.6. *The functor sending $F \in \mathcal{SH}(k)^{c\heartsuit}[1/e]$ to the presheaf on $PM_{MGL}(k)$ given by $F_*(X) = [\Sigma 2*, *X_+, F \wedge MGL]$ induces an equivalence of categories between $\mathcal{SH}(k)^{c\heartsuit}[1/e]$ and the category of enriched presheaves on $PM_{MGL}(k)$ (with values in $MU_{2*}MU$-comodules).*

The proof of this theorem is finished by Remark 4.11, which is the equivalence between $MGL_{c=0}\text{-}Mod$ and $\mathcal{P}_{\mathcal{SH}}(PM_{MGL}(k))$ induced by their heart. Then,

$$F : \mathbb{1}_{c=0}\text{-}Mod_b \leftrightarrow MGL_{c-0}\text{-}Mod_b \simeq D^b(MGL_{c-0}\text{-}Mod_b^{c\heartsuit}) : R. \tag{33}$$

Then check according to Lurie's theorem. i.e. F, R t-exact, $MGL_{c-0}\text{-}Mod_b$ has enough injective and so on.

4.3 W-cellular objects

To make our result suitable for some explicit calculation, we still need to restrict our view on the cellular case. We have a generalization of cellular objects which are called W-cellulars.

Definition 4.2. Let W be a set of smooth proper schemes over k that contains $Spec(k)$ and is closed under finite products. Define the W-cellular category, denoted by $\mathcal{SH}(k)^{wcell}$ to be the subcategory of $\mathcal{SH}(k)$ generated under taking colimits and desuspensions by objects of the form $Th(\xi)$ for $\xi \in K(X)$ and $X \in W$.

Then we can get W-cellular Chow t-structure by "restrict" $Th(\xi)$ for $\xi \in K(X)$ and $X \in W$ and they share similar properties with the cellular case. What's more, the cellularization functor is t-exact for the Chow t-structures as well as the $A\text{-}Mod$, $A^{wcell}\text{-}Mod$. The reconstruction theorem also has a W-cellular version. Since we have little time, we will omit details.

The most interesting application of W-cellular is shown as follows.

First, set W to be $\{Spec(l) | l/k$ is a finite separable extension $\}$. Let $G = Gal(k)$ be the absolute Galois group. Recall that the stable category of genuine G-spectra $\mathcal{SH}(BG)$ [BH17, Example 9.12] admits a t-structure with heart the category of G-Mackey functors [BH17, Proposition 9.11]

$$\mathcal{P}_{Ab}(Span(Fin_G)) =: Mack_G \tag{34}$$

Here Fin_G denotes the category of finite discrete G-sets. The canonical symmetric monoidal cocontinuous functor $\mathcal{SH} \to \mathcal{SH}(BG)$ induces the "constant Mackey functor"

$$Ab \to Mack_G, A \to \underline{A}. \tag{35}$$

This is symmetric monoidal and so preserves rings, Hopf algebroids etc. In particular, from the usual Hopf algebroid $(MU_{2*}, MU_{2*}MU)$ we obtain the constant Hopf algebroid in (graded) Mackey functors $(\underline{MU_{2*}}, \underline{MU_{2*}MU})$.

As a result, we have such equivalence:

Corollary. *Set W to be* $\{Spec(l)|l/k$ *is a finite separable extension* $\}$. *Let* $G = Gal(k)$ *be the absolute Galois group. We have*

$$\mathcal{SH}(k)^{wcell,c\heartsuit} \simeq \underline{MU_{2*}MU}\text{-}CoMod \tag{36}$$

$$\mathbb{1}_{c=0}\text{-}Mod^{wcell} \simeq Hov(\underline{MU_{2*}MU}) \tag{37}$$

$$MGL_{c=0}\text{-}Mod^{wcell,c\heartsuit} \simeq \underline{MU_{2*}}\text{-}Mod. \tag{38}$$

Here $\text{-}CoMod$, Hov *and* $\text{-}Mod$ *are performed relative to* $Mack_G$.

For the trivial extension, we can remove the underline, and then we get the theorem 1.9, which supports us to do some explicit calculations with these equivalences. The idea of the proofs is similar to the normal case so I will not write it again.