Notes of GWX

Zhonglin Wu
SUSTech

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1 Introduction

In this note, we suppose everything is $H\mathbb{F}_p^{mot}$-completed. $H\mathbb{F}_p^{mot}{}_{*,*} = \mathbb{F}_p[\tau]$ where τ is in bidegree $(0,-1)$. $\widehat{S^{1,0}}$ is the suspension spectrum of the simplicial sphere, and $\widehat{S^{1,1}}$ is the suspension spectrum of the multiplicative group.

The main result of GWX is the equivalence between two ∞-categories with t-structure.

Theorem 1.1. *There is an equivalence of stable ∞-categories equipped with t-structures at each prime p:*

$$\mathcal{D}^b(BP_*BP\text{-}Comod^{ev}) \simeq \widehat{S^{0,0}}/\tau\text{-}Mod^b_{harm} \tag{1}$$

between the bounded derived category of p-completed BP_*BP-comodules that are concentrated in even degrees, and the category of harmonic motivic left module spectra over $\widehat{S^{0,0}}/\tau$, whose MGL-homology has bounded Chow-Novikov degree, with morphisms the $\widehat{S^{0,0}}/\tau$-linear map.

This equivalence induces an equivalence in the motivic Adams spectral sequence and the algebra Novikov spectral sequence.

Theorem 1.2. *For each prime p, there is an isomorphism of spectral sequences between the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$ and the algebraic Novikov spectral sequence for the classical sphere spectrum $\widehat{S^0}$. Here τ can be lifted to a map $\widehat{S^{0,-1}} \to \widehat{S^{0,0}}$, and $\widehat{S^{0,0}}/\tau$ is its cofiber.*

As a result, we have

Theorem 1.3. *There is an isomorphism between $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ and $\pi_{*,*}(\widehat{S^{0,0}}/\tau)$ that preserves the multiplicative filtrations, composition products and higher compositions in the respective categories.*

With this equivalence, some differentials in the classical Adams spectral sequence can be detected by the algebraic Novikov spectral sequence. In addition, this equivalence is related to the Miller square.

We can build a motivic version Miller square. The classical Miller square is a diagram of spectral sequences. Theorem Adams spectral sequence and the Adams-Novikov spectral sequence are connected by this diagram. By replacing these spectral sequences with the motivic version, we can get the motivic Miller square. Since some differentials can be detected by the classical Miller square, the same method can be used to detect differentials for the motivic version.

What's more, the spectral sequences in the classical Miller square are isomorphic to the motivic version except the Adams spectral sequence and the motivic Adams spectral sequence. However, they are isomorphic if we invert the τ-torsion part of the motivic Adams spectral sequence.

To compute a nontrivial classical Adams differential, for any r, start with an algebraic Novikov d_r-differential. Theorem 1.17 gives us a motivic Adams d_r-differential for $\widehat{S^{0,0}}/\tau$. Pulling back to the bottom cell of $\widehat{S^{0,0}}/\tau$ of the source element gives us a motivic Adams $d_{r'}$-differential for the motivic sphere with $r' \leq r$. Using the Betti realization functor, we then obtain a classical Adams $d_{r'}$-differential!

2 An algebraic model for cellular MU^{mot}/τ-modules.

Definition 2.1. Denote by

$$MU^{mot}_{*,*}/\tau\text{ - }Mod \tag{2}$$

the abelian category of graded left modules over $MU^{mot}_{*,*}/\tau$, and by

$$MU^{mot}_{*,*}/\tau\text{ - }Mod^0 \tag{3}$$

as the full subcategory of $MU^{mot}_{*,*}/\tau\text{ - }Mod$ spanned by all graded modules $M_{*,*}$ that are concentrated in Chow-Novikov degree 0,

The relation between $MU^{mot}/\tau\text{ - }Mod^b_{cell}$ and $\mathcal{D}^d(MU_*\text{ - }Mod^{ev})$ can be seen as a simpler version of the relation between $\widehat{S^{0,0}}/\tau\text{ - }Mod^b_{harm}$ and $\mathcal{D}^b(MU_*MU\text{ - }Comod^{ev})$. So section 4 in GWX shares the same structure as section 3 in the original paper, and we can start with the simple case. Besides as a preview of the following section 4, the result in this chapter is also helpful to the construction of MU^{mot}/τ-based Adams resolutions in the category $\widehat{S^{0,0}}/\tau\text{ - }Mod^b_{cell}$.

We want to show that there is a t-exact equivalence of stable ∞-categories:

$$MU^{mot}/\tau\text{ - }Mod^b_{cell} \to \mathcal{D}^d(MU_*\text{ - }Mod^{ev}), \tag{4}$$

whose restriction on the heart is given by

$$MU^{mot}/\tau\text{ - }Mod^{\heartsuit}_{cell} \to MU_*\text{ - }Mod^{ev}. \tag{5}$$

To prove the above equivalence, we will use the following strategy. We start with building a spectral sequence which is called the universal coefficient spectral sequence in $MU^{mot}/\tau\text{ - }Mod_{cell}$. With the help of this spectral sequence, we can prove the equivalence on the heart for the given t-structure on $MU^{mot}/\tau\text{ - }Mod^b_{cell}$. The equivalence can be extended to the whole ∞-categories by Lurie's theorem.

First, We have an equivalence $MU^{mot}_{*,*}/\tau\text{ - }Mod \cong MU_*\text{ - }Mod^{ev}$ which is induced by $\pi_{*,*}$. The sketch of the proof is listed as follows. We can prove it by the universal coefficient spectral sequence as the theorem

3.2.

$$E_2^{s,t,w} = Ext^{s,t,w}_{MU^{mot}_{*,*}/\tau}(\pi_{*,*}(X), \pi_{*,*}(Y)) \Rightarrow [\Sigma^{t-s,w}X, Y]_{MU^{mot}/\tau}. \tag{6}$$

The main idea of the proof is to consider a free resolution of $\pi_{*,*}X$ over $MU^{mot}_{*,*}/\tau$ and build the spectral sequence in a classical way.

To be specific, there is an isomorphism $[X,Y]_{MU^{mot}/\tau} \to Hom_{MU^{mot}_{*,*}/\tau}(\pi_{*,*}(X), \pi_{*,*}(Y))$ induced by $\pi_{*,*}$ if we give some restriction on X and Y which allows us compute homotopy groups purely algebraically. Considering the possible targets and sources of $E_2^{0,0,0}$, which are vanishing due to there being nothing in the possible range, is the main idea of the proof.

Next, we will prove the equivalence of the heart. $\pi_{*,*}$ induces this functor, which is fully faithful according to the above result. The surjective is proved in Proposition 3.5. To achieve this target, we can realize the free object at first, then for normal objects, find a free resolution, and then build an Adams-like tower. The colimit of those fibers is the realization in $MU^{mot}/\tau\text{-}Mod^{\heartsuit}_{cell}$.

Then we need to show the natural filtrations (by the Chow-Novikov degree) form a t-structure on $MU^{mot}_{*,*}/\tau\text{-}Mod^{b}_{cell}$. We just need to check those four properties in Proposition 3.6, which is a routine. Finally, we just need to show that the homotopy groups vanish at the negative degrees. That's what we do in theorem 3.8. That's also completed due to the universal coefficient spectral sequence due to the degree reason. So we have $MU^{mot}/\tau\text{-}Mod^{b}_{cell} \to \mathcal{D}^d(MU_*\text{-}Mod^{ev})$ as an equivalence between ∞-categories.

3 An algebraic model for harmonic $\widehat{S^{0,0}}/\tau$-modules.

The main target of this section is to prove that there is a t-exact equivalence of stable ∞-categories

$$\widehat{S^{0,0}}/\tau\text{-}Mod^{b}_{harm} \to \mathcal{D}^b(MU_*MU\text{-}Comod^{ev}), \tag{7}$$

whose restriction on the heart is given by

$$MU^{mot}_{*,*} : \widehat{S^{0,0}}/\tau\text{-}Mod^{\heartsuit}_{harm} \to MU_*MU\text{-}Comod^{ev}. \tag{8}$$

Here we will give a brief introduction of the category of harmonic $\widehat{S^{0,0}}/\tau$-modules and the category of MU_*MU-comodules at first. A $\widehat{S^{0,0}}/\tau$-module spectrum Y is harmonic if it is $\widehat{S^{0,0}}/\tau$-cellular and the natural map $Y \to Y^{\wedge}_{MU^{mot}}$ is an isomorphism on $\pi_{*,*}$. Since the two completions $X^{\wedge}_{MGL}$ in the category $\mathbb{C}-mot-Spectra$ and $X^{\wedge}_{MU^{mot}}$ in the category $\widehat{S^{0,0}}\text{-}Mod$ are equivalent for any X in $\widehat{S^{0,0}}\text{-}Mod_{cell}$. [1] The category of harmonic $\widehat{S^{0,0}}/\tau$-module spectra is denoted by $\widehat{S^{0,0}}\text{-}Mod_{harm}$, which is a subcategory of $\widehat{S^{0,0}}\text{-}Mod_{cell}$.

If we forget the motivic weight, we have the equivalence

$$MU_*MU\text{-}Comod^{ev} \cong MU^{mot}_{*,*}MU^{mot}/\tau\text{-}Comod^{0}, \tag{9}$$

so we can get a diagram like the above section.

[1] Being harmonic is closed under taking suspensions, finite products and fibers.

The strategy of proof is similar to the strategy of the above section but some techniques are introduced to overcome some new problems. [2]

The first problem is that we need to change $\pi_{*,*}$ to $MU^{mot}_{*,*}$, the MU-homology. As a result, the universal coefficient spectral sequence is replaced by a new spectral sequence called the absolute Adams-Novikov spectral sequence which is introduced in section 5.

The second problem appears at the equivalence between the hearts. To be specific, since we don't have the free resolution for the part of essential surjectivity. We need the Landweber Filtration Theorem to construct the preimage.

Landweber's Filtration Theorem (Theorem 4.4) describes comodules over MU_*MU "filtrational". That is, any comodule M over MU_*MU whose underlying MU_*-module is finitely presented, can be reconstructed by finitely many extensions of suspensions of MU_*/I_n.

Then we need to find the preimage of MU_*/I_n. It is proved by the 2-out-of-3 lemma (along a short exact sequence) and the induction on n for MU_*/I_n. With the lemma, we just need to consider the preimage of $MU^{mot}_{*,*}$ (Prop 4.11), and it has been proved in the above section.

Finally, we still need to show that the t-structure acts as we expect as well as the equivalence between ∞-categories. That is a routine.

3.1 Absolute Adams-Novikov spectral sequence

We need this spectral sequence since the classical one leads us to the $\pi_{*,*}Y^{\wedge}_{MU^{mot}}$ instead of $[X, Y^{\wedge}_{MU^{mot}}]_{\widehat{S^{0,0}}/\tau}$. We have two reasons for that. The first reason is we are dealing with morphisms in $\widehat{S^{0,0}}/\tau\text{-}Mod_{cell}$ instead of classical motivic spectra. Another reason is we need a general cellular $\widehat{S^{0,0}}/\tau$-module than just the unit object $\widehat{S^{0,0}}/\tau$ for the position of X. The relative injective resolution also fails due to the X may not be a projective object. So we need the absolute one.

For the absolute Adams-Novikov spectral sequence, we resolute with "absolute" injective objects instead of relative injectives. Lemma 5.1 and 5.3 show the relation between an injective module in $MU^{mot}_{*,*}/\tau\text{-}Mod^0$ and its realization in $MU^{mot}/\tau\text{-}Mod^{\heartsuit}_{cell}$. Those lemmas show that their preimage of the injective objects in $MU^{mot}_{*,*}/\tau\text{-}Mod^0$ are injective, and those injective objects have injective images in $MU^{mot}_{*,*}/\tau\text{-}Mod^0$ satisfy our demands (as Y). Then we need to resolute ordinary Y by those objects that have injective images in $MU^{mot}_{*,*}/\tau\text{-}Mod^0$. This resolution is the absolute Adams-Novikov resolution. [3]

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[2]The connections between the categories in this section and the previous section are shown in the second diagram of P39.

[3]Of course, we need to prove that such a resolution exists. That's what we do in Prop 5.5. In some views, the category $MU_*MU\text{-}Comod^{ev}$ has enough injective objects. Due to this fact, we can always find such a resolution. What's more, in the proof, we need to construct a map between this tower and the classical Adams-Novikov tower, that is useful in the proof of the convergence of this spectral sequence (Theorem 5.6).

4 The equivalence between spectral sequence.

Since we already have t-exact equivalences between some ∞-categories, Here's a natural question: when does such equivalence induce an equivalence between spectral sequences? In this section, we will show the gap and list those things we need to check.

4.1 The algebraic Novikov tower

For the BP_*BP-comodule BP_*, its relative injective resolution

$$BP_* \to C_0^0 \to C_0^1 \to \cdots \tag{10}$$

is a long exact sequence in the abelian category of BP_*BP-comodules, that satisfies the following two conditions:

- The long exact sequence (9.1) is split exact as BP_*-modules.
- Each comodule C_0^s is relative injective.

From now on, we fix such a relative injective resolution C_0^* of BP_* that is concentrated in even internal degrees. Such a relative injective resolution exists. [4].

For $a \geq 1$ let C_a^* be the sub cochain complex of C_0^* defined by

$$C_a^s = I^{a-s} C_0^s \tag{11}$$

Here $I^r = BP_*$ for $r \leq 0$.

Denote Q_a^* as the quotient cochain complex of the inclusion map $C_{a+1}^* \to C_a^*$, we have a tower of cochain complexes as follows:

Here's the diagram

Here $Q_a^s = I^{a-s}C_0^s / I^{a-s+1}C_0^s$ which is bounded by definition.

Therefore, although the cochain complexes C_a^* are unbounded, they live in the category $\mathcal{D}^b(BP_*BP - Comod^{ev})$.

Applying the functor

$$R^{*,*}Hom_{BP_*BP}(BP_*, -), \tag{12}$$

where $R^{*,*}Hom_{BP_*BP}(BP_*, -)$ is the derived homomorphisms in the category $\mathcal{D}^b(BP_*BP - Comod^{ev})$, we get a spectral sequence with the E_1-page

$$R^{*,*}Hom_{BP_*BP}(BP_*, Q_a^*) \tag{13}$$

converging to

$$R^{*,*}Hom_{BP_*BP}(BP_*, BP_*) = Ext^{*,*}_{BP_*BP}(BP_*, BP_*). \tag{14}$$

This is the regraded algebraic Novikov spectral sequence.

[4] for example the cobar complex

4.2 Characterization of Adams towers

We denote:

$$H\mathbb{F}_p^{mot}/\tau := \widehat{S^{0,0}}/\tau \wedge_{\widehat{S^{0,0}}} H\mathbb{F}_p^{mot} \tag{15}$$

Definition 4.1. A tower in $\widehat{S^{0,0}}/\tau - Mod^b_{harm}$ is a motivic Adams tower if

Here's the diagram

- Each motivic spectrum K_m is a retract of a wedge of suspensions of $H\mathbb{F}_p^{mot}/\tau$
- Each map $f_m : X_m \to K_m$ induces an epimorphism on the $H\mathbb{F}_p^{mot}$-cohomology. Or equivalently, each map $g_m : X_{m+1} \to X_m$ induces the zero map on the $H\mathbb{F}_p^{mot}$-cohomology.

Actually the second condition equivalent to check that each map g_m induces the zero map on $[-, H\mathbb{F}_p^{mot}/\tau]_{\widehat{S^{0,0}}/\tau}$ Since

$$\mathcal{D}^b(BP_*BP - Comod^{ev}) \simeq \mathcal{D}^b(MU_*MU - Comod^{ev}) \simeq \widehat{S^{0,0}}/\tau - Mod^b_{harm}, \tag{16}$$

we can pull the first tower into the category of $\widehat{S^{0,0}}/\tau - Mod^b_{harm}$.

Proposition 4.1. *The above tower is a motivic Adams tower in the sense of the above definition, if the following two conditions are satisfied for the re-graded algebraic Novikov tower in the category $\mathcal{D}^b(BP_*BP-Comod^{ev})$:*

- *Each Q_a^* is quasi-isomorphic to a retract of a direct sum of shifts of BP_*BP/I.*
- *Each map $q_a : C_a^* \to Q_a^*$ induces an epimorphism on $R^{*,*}Hom_{BP_*}(-, \mathbb{F}_p)$. Or equivalently, each map $i_a : C_{a+1}^* \to C_a^*$ induces the zero map on $R^{*,*}Hom_{BP_*}(-, \mathbb{F}_p)$.*

These conditions correspond to those conditions in that definition.

The first condition stands due to such lemma:

Lemma 4.2. *Under the equivalences of the hearts, $H\mathbb{F}_p^{mot}/\tau$ corresponds to BP_*BP/I.*

The second one stands due to such lemma:

Lemma 4.3. *Suppose that X is in the category $\widehat{S^{0,0}}/\tau - Mod^b_{harm}$ and that $C^*(X)$ is the cochain complex of BP_*BP-comodules representing the image of X under the equivalence between ∞-categories. Then we have*

$$[\Sigma^{*,*}X, H\mathbb{F}_p^{mot}/\tau]_{\widehat{S^{0,0}}/\tau} \cong R^{*,*}Hom_{BP_*}(C^*(X), \mathbb{F}_p) \tag{17}$$

where $R^{*,*}Hom_{BP_*}(-,-)$ is the derived homomorphism in the derived category of BP_*-modules.